Solutions to the MATH 118 midterm test.

1. (10pts total; 5 pts each subitem) Suppose $K(x,y)$ is a continuous compactly supported function on $\mathbb{R}\times\mathbb{R}$. Define $T: L^2(\mathbb{R})\to L^2(\mathbb{R})$ by

$$T(f)(x) := \int_{y\in\mathbb{R}} f(y)K(x,y)\,dy.$$

Show that T is a linear map and that its adjoint is given by

$$T^*(g)(x) := \int_{y\in\mathbb{R}} g(y)\overline{K(y,x)}\,dy.$$

Solution. The map T is linear, since it respects linear combinations of functions:

$$\begin{aligned} T(\alpha f+\beta g)(x) &= \int_{y\in\mathbb{R}} (\alpha f(y)+\beta g(y))K(x,y)\,\mathrm{dy} = \alpha\int_{y\in\mathbb{R}} f(y)K(x,y)\,\mathrm{dy} \\ &\quad +\beta\int_{y\in\mathbb{R}} g(y)K(x,y)\,\mathrm{dy} = \alpha(Tf)(x)+\beta(Tg)(x). \end{aligned}$$

To find the adjoint, we form the inner product of Tf and g (where both f and g are arbitrary), change the order of integration, and rename variables of integration:

$$\begin{aligned} \langle Tf, g\rangle &= \int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} f(y)K(x,y)\,\mathrm{dy}\overline{g(x)}\,\mathrm{dx} = \int_{y\in\mathbb{R}} f(y)\overline{\int_{x\in\mathbb{R}} \overline{K(x,y)}g(x)\,\mathrm{dx}} \\ &= \int_{x\in\mathbb{R}} f(x)\overline{\int_{y\in\mathbb{R}} \overline{K(y,x)}g(y)\,\mathrm{dy}\,\mathrm{dx}} = \langle f, T^*g\rangle. \end{aligned}$$

Since f and g are arbitrary, this implies that $(T^*g)(x) = \int_{y\in\mathbb{R}} \overline{K(y,x)}g(y)\,\mathrm{dy}$.

2. (10pts total) Show that the sinc function

$$\mathrm{sinc}(x) := \begin{cases} \frac{\sin(\pi x)}{\pi x} & x\neq 0 \\ 1 & x=0 \end{cases}$$

solves the refinement equation

$$\phi(x) = \phi(2x) + \sum_{k\in\mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi}\phi(2k-2k-1).$$

Solution. Recall that the Fourier transform of the characteristic function $\chi_{[-\pi,\pi]}$ of the interval $[-\pi,\pi]$ is (almost) the sinc function:

$$\int_{-\pi}^{\pi} e^{-\mathrm{i}\lambda x}\,\mathrm{dx} = \left.\frac{e^{-\mathrm{i}\lambda x}}{-\mathrm{i}\lambda}\right|_{x=-\pi}^{x=\pi} = 2\pi\frac{\sin(\pi\lambda)}{\pi\lambda}.$$

Stretching this function and modulating it (multiplying by exponentials) will produce the desired shifts

$$\frac{1}{2}\int_{-2\pi}^{2\pi} e^{-\mathrm{i}\lambda x+\mathrm{i}kx/2}\,\mathrm{dx} = 2\pi\frac{\sin(\pi(2\lambda-k))}{\pi(2\lambda-k)}.$$

Thus we need to solve the following equation on the Fourier domain:

$$\chi_{[-\pi,\pi]}(\lambda) = \chi_{[-2\pi,2\pi]}(\lambda)\sum_{k\in\mathbb{Z}}\frac{p_k}{2}e^{\mathrm{i}k\lambda/2}.$$

So, we must find the Fourier series for $\chi_{[-\pi,\pi]}$ on the interval $[-2\pi, 2\pi]$. Since

$$\int_{-\pi}^{\pi} e^{-\mathrm{i}k\lambda/2}d\lambda = \begin{cases} 2\pi & k=0 \\ \frac{4\sin(k\pi/2)}{k} = \begin{cases} 0 & k \text{ is even} \\ \frac{4(-1)^{(k-1)/2}}{k} & k \text{ is odd} \end{cases} & k\neq 0, \end{cases}$$

we get

$$\chi_{[-\pi,\pi]}(\lambda) = \chi_{[-2\pi,2\pi]}(\lambda)\left(\frac{1}{2}+\sum_{k\in\mathbb{Z}}\frac{(-1)^k}{2k+1}e^{\mathrm{i}(2k+1)\lambda/2}\right),$$

hence

$$\phi(x) = \phi(2x) + \sum_{k\in\mathbb{Z}}\frac{2(-1)^k}{(2k+1)\pi}\phi(2x-2k-1).$$

3. (12pts total; 2pts for each subitem) Are the following filters (a) linear (b) time-invariant (c) causal?

1. $(L_1 f)(t) = f(t) - \int_t^{t^2} f(x)\,\mathrm{dx}$,
2. $(L_2 f)(t) := \int_{-\infty}^{\infty} f(x)e^{-(x-t)^2}\,\mathrm{dx}$.

Solution. The first filter is linear, not time-invariant, and not causal. Indeed, it is a linear combination of the identity map and an integration operator, and both are linear, so the filter is linear as well. It is not time-invariant, which can be seen by applying it to a shift f_a of an arbitrary function f:

$$\begin{aligned}(L_1 f_a)(t) &= f_a(t) - \int_t^{t^2} f)a(x)\,\mathrm{dx} = f(t-a) - \int_t^{t^2} f(x-a)\,\mathrm{dx} \\ &= f(t-a) - \int_{t-a}^{t^2-a} f(x)\,\mathrm{dx} \neq f(t-a) - \int_{t-a}^{(t-a)^2} f(x)\,\mathrm{dx} = (L_1 f)(t-a).\end{aligned}$$

L_1 is not causal, which can be checked, for example, by applying L_1 to the function f which is 0 over $\mathbb{R}_-$ and 1 over $\mathbb{R}_+$: for $t<0$, we get

$$(L_1 f)(t) = -\int_0^{t^2}\,\mathrm{dx} = -t^2 \neq 0.$$

The second filter is linear and time-invariant, since it is given explicitly as a convolution with the function e^{-x^2}. Since the support of this function is all of $\mathbb{R}$, the filter is not causal (causality requires that the support be a subset of $\mathbb{R}_+$).

4. (8pts total) Let ϕ and ψ be the Haar scaling function and the Haar wavelet, respectively. We denote, as usual,

$$V_j := \text{span}\{\phi(2^j \cdot - k), \ \ k \in \mathbb{Z}\}, \qquad W_j := \text{span}\{\psi(2^j \cdot - k), \ \ k \in \mathbb{Z}\}.$$

Express the function f given by

$$f(x) := \begin{cases} 2 & 0 \le x \le 1/4, \\ -3 & 1/4 \le x < 1/2, \\ 1 & 1/2 \le x < 3/4, \\ 3 & 3/4 \le x < 1, \\ 0 & \text{otherwise} \end{cases}$$

in terms of its components in V_0, W_0 and W_1.

Solution. We are given coefficients at level 2, i.e,, $a^{[2]} = [2, -3, 1, 3]$. Using the wavelet decomposition algorithm for the Haar system, we get

$$\begin{aligned} a_0^{[1]} &= \frac{a_0^{[2]} + a_1^{[2]}}{2} = -\frac{1}{2}, \quad & a_1^{[1]} &= \frac{a_2^{[2]} + a_3^{[2]}}{2} = 2, \\ b_0^{[1]} &= \frac{a_0^{[2]} - a_1^{[2]}}{2} = \frac{5}{2}, \quad & b_1^{[1]} &= \frac{a_2^{[2]} - a_3^{[2]}}{2} = -1, \\ a_0^{[0]} &= \frac{a_0^{[1]} + a_1^{[1]}}{2} = \frac{3}{4}, \quad & b_0^{[0]} &= \frac{a_0^{[1]} - a_1^{[1]}}{2} = -\frac{5}{4}. \end{aligned}$$

So, the representation of f in terms of its wavelet components from W_0 and W_1 and its coarse component from V_0 is

$$f(x) = \underbrace{\frac{3}{4}\phi(x)}_{\in V_0} - \underbrace{\frac{5}{4}\psi(x)}_{\in W_0} + \underbrace{\frac{5}{2}\psi(2x) - \psi(2x-1)}_{\in W_1}.$$