Solutions to the MATH 118 midterm test.

1. (10pts total; 5 pts each subitem) Suppose \( K(x, y) \) is a continuous compactly supported function on \( \mathbb{R} \times \mathbb{R} \). Define \( T : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) by

\[
T(f)(x) := \int_{y \in \mathbb{R}} f(y) K(x, y) \, dy.
\]

Show that \( T \) is a linear map and that its adjoint is given by

\[
T^*(g)(x) := \int_{y \in \mathbb{R}} g(y) \overline{K(y, x)} \, dy.
\]

**Solution.** The map \( T \) is linear, since it respects linear combinations of functions:

\[
T(\alpha f + \beta g)(x) = \int_{y \in \mathbb{R}} (\alpha f(y) + \beta g(y)) K(x, y) \, dy = \alpha \int_{y \in \mathbb{R}} f(y) K(x, y) \, dy + \beta \int_{y \in \mathbb{R}} g(y) K(x, y) \, dy = \alpha (Tf)(x) + \beta (Tg)(x).
\]

To find the adjoint, we form the inner product of \( Tf \) and \( g \) (where both \( f \) and \( g \) are arbitrary), change the order of integration, and rename variables of integration:

\[
\langle Tf, g \rangle = \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} f(y) K(x, y) \, dy \overline{g(x)} \, dx = \int_{y \in \mathbb{R}} f(y) \int_{x \in \mathbb{R}} \overline{K(x, y)} g(x) \, dx = \int_{x \in \mathbb{R}} f(x) \int_{y \in \mathbb{R}} K(y, x) g(y) \, dy \, dx = \langle f, T^* g \rangle.
\]

Since \( f \) and \( g \) are arbitrary, this implies that \( (T^* g)(x) = \int_{y \in \mathbb{R}} \overline{K(y, x)} g(y) \, dy \).

2. (10pts total) Show that the sinc function

\[
sinc(x) := \begin{cases} \frac{\sin(x \pi)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}
\]

solves the refinement equation

\[
\phi(x) = \phi(2x) + \sum_{k \in \mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi} \phi(2k - 2x - 1).
\]

**Solution.** Recall that the Fourier transform of the characteristic function \( \chi_{[-\pi, \pi]} \) of the interval \( [-\pi, \pi] \) is (almost) the sinc function:

\[
\mathcal{F}(\chi_{[-\pi, \pi]})(x) = \frac{\sin(\pi x)}{\pi x}.
\]

Then, for any bounded and piecewise smooth function \( \phi \), we have

\[
\phi(x) = \sum_{k \in \mathbb{Z}} \phi(2k - 2x - 1).
\]

To see this, we can use the Parseval's identity

\[
\int_{-\pi}^{\pi} \phi(x) \, dx = \int_{-\pi}^{\pi} \mathcal{F}(\phi)(x) \, dx.
\]

Since \( \mathcal{F}(\phi)(x) = \sum_{k \in \mathbb{Z}} \phi(2k - 2x - 1) \), we have

\[
\int_{-\pi}^{\pi} \phi(x) \, dx = \sum_{k \in \mathbb{Z}} \int_{-\pi}^{\pi} \phi(2k - 2x - 1) \, dx.
\]

This implies that \( \phi(x) = \sum_{k \in \mathbb{Z}} \phi(2k - 2x - 1) \) almost everywhere.
Stretching this function and modulating it (multiplying by exponentials) will produce the desired shifts
\[ \frac{1}{2} \int_{-2\pi}^{2\pi} e^{-i\lambda x + ikx/2} \, dx = \frac{2\pi}{\pi(2\lambda - k)} \sin(\pi(2\lambda - k)). \]

Thus we need to solve the following equation on the Fourier domain:
\[ \chi[-\pi,\pi](\lambda) = \chi[-2\pi,2\pi](\lambda) \sum_{k \in \mathbb{Z}} \frac{p_k}{2} e^{ik\lambda/2}. \]

So, we must find the Fourier series for \( \chi[-\pi,\pi] \) on the interval \([-2\pi, 2\pi]\). Since
\[ \int_{-\pi}^{\pi} e^{-ik\lambda/2} \, d\lambda = \left\{ \begin{array}{ll} 2\pi & \text{if } k = 0 \\ \frac{4\sin(k\pi/2)}{k} & \text{if } k \text{ is even} \\ \frac{4(-1)^{(k-1)/2}}{k} & \text{if } k \neq 0, k \text{ is odd} \end{array} \right. \]

we get
\[ \chi[-\pi,\pi](\lambda) = \chi[-2\pi,2\pi](\lambda) \left( \frac{1}{2} + \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{2k + 1} e^{i(2k+1)\lambda/2} \right), \]

hence
\[ \phi(x) = \phi(2x) + \sum_{k \in \mathbb{Z}} \frac{2(-1)^k}{(2k + 1)\pi} \phi(2x - 2k - 1). \]

3. (12pts total; 2pts for each subitem) Are the following filters (a) linear (b) time-invariant (c) causal?

1. \((L_1 f)(t) = f(t) - \int_t^2 f(x) \, dx,\)
2. \((L_2 f)(t) := \int_{-\infty}^\infty f(x)e^{-(x-t)^2} \, dx.\)

**Solution.** The first filter is linear, not time-invariant, and not causal. Indeed, it is a linear combination of the identity map and an integration operator, and both are linear, so the filter is linear as well. It is not time-invariant, which can be seen by applying it to a shift \( f_a \) of an arbitrary function \( f \):
\[
(L_1 f_a)(t) = f_a(t) - \int_t^2 f(x - a) \, dx = f(t) - \int_t^2 f(x - a) \, dx = f(t - a) - \int_t^{t-a} f(x) \, dx \\
= f(t - a) - \int_{t-a}^{t-a} f(x) \, dx \neq f(t - a) - \int_{t-a}^{t-a} f(x) \, dx = (L_1 f)(t - a).
\]

\( L_1 \) is not causal, which can be checked, for example, by applying \( L_1 \) to the function \( f \) which is 0 over \( \mathbb{R}_- \) and 1 over \( \mathbb{R}_+ \): for \( t < 0 \), we get
\[
(L_1 f)(t) = - \int_{t}^{0} dx = -t^2 \neq 0.
\]
The second filter is linear and time-invariant, since it is given explicitly as a convolution with the function $e^{-x^2}$. Since the support of this function is all of $\mathbb{R}$, the filter is not causal (causality requires that the support be a subset of $\mathbb{R}_+$).

4. (8pts total) Let $\phi$ and $\psi$ be the Haar scaling function and the Haar wavelet, respectively. We denote, as usual,

$$V_j := \text{span}\{\phi(2^j \cdot -k), \ k \in \mathbb{Z}\}, \quad W_j := \text{span}\{\psi(2^j \cdot -k), \ k \in \mathbb{Z}\}.$$  

Express the function $f$ given by

$$f(x) := \begin{cases} 
2 & 0 \leq x \leq 1/4, \\
-3 & 1/4 \leq x < 1/2, \\
1 & 1/2 \leq x < 3/4, \\
3 & 3/4 \leq x < 1, \\
0 & \text{otherwise}
\end{cases}$$

in terms of its components in $V_0$, $W_0$ and $W_1$.

**Solution.** We are given coefficients at level 2, i.e., $a^{[2]} = [2, -3, 1, 3]$. Using the wavelet decomposition algorithm for the Haar system, we get

$$a_0^{[1]} = \frac{a_0^{[2]} + a_1^{[2]}}{2} = -\frac{1}{2}, \quad a_1^{[1]} = \frac{a_2^{[2]} + a_3^{[2]}}{2} = 2,$$

$$b_0^{[1]} = \frac{a_0^{[2]} - a_1^{[2]}}{2} = \frac{5}{2}, \quad b_1^{[1]} = \frac{a_2^{[2]} - a_3^{[2]}}{2} = -1,$$

$$a_0^{[0]} = \frac{a_0^{[1]} + a_1^{[1]}}{2} = \frac{3}{4}, \quad b_0^{[0]} = \frac{a_0^{[1]} - a_1^{[1]}}{2} = -\frac{5}{4}.$$  

So, the representation of $f$ in terms of its wavelet components from $W_0$ and $W_1$ and its coarse component from $V_0$ is

$$f(x) = \underbrace{\frac{3}{4} \phi(x)}_{\in V_0} - \underbrace{\frac{5}{4} \psi(x)}_{\in W_0} + \underbrace{\frac{5}{2} \psi(2x)}_{\in W_1} - \psi(2x - 1).$$