MATH 118, midterm test, Oct 21st.

Name
Student ID #

All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Partial credit **may** be given but only for significant progress towards a solution. Show all relevant work in logical sequence and indicate all answers clearly. Cross out all work you do not wish considered. Books and notes are allowed. Calculators, computers, cell phones, pagers and similar devices are not allowed during the test.

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1. (10pts total; 5 pts each subitem) Suppose $K(x,y)$ is a continuous compactly supported function on $\mathbb{R} \times \mathbb{R}$. Define $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$T(f)(x) := \int_{y\in\mathbb{R}} f(y)K(x,y)\,dy.$$

Show that T is a linear map and that its adjoint is given by

$$T^*(g)(x) := \int_{y\in\mathbb{R}} g(y)\overline{K(y,x)}\,dy.$$

2. (10pts total) Show that the sinc function

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

solves the refinement equation

$$\phi(x) = \phi(2x) + \sum_{k \in \mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi} \phi(2k - 2k - 1).$$

3. (12pts total; 2pts for each subitem) Are the following filters (a) linear (b) time-invariant (c) causal?

1. $(L_1 f)(t) = f(t) - \int_t^{t^2} f(x)\mathrm{dx}$,
2. $(L_2 f)(t) := \int_{-\infty}^{\infty} f(x) e^{-(x-t)^2}\mathrm{dx}$.

4. (8pts total) Let ϕ and ψ be the Haar scaling function and the Haar wavelet, respectively. We denote, as usual,

$$V_j := \mathrm{span}\{\phi(2^j \cdot -k),\ \ k \in \mathbb{Z}\}, \qquad W_j := \mathrm{span}\{\psi(2^j \cdot -k),\ \ k \in \mathbb{Z}\}.$$

Express the function f given by

$$f(x) := \begin{cases} 2 & 0 \leq x \leq 1/4, \\ -3 & 1/4 \leq x < 1/2, \\ 1 & 1/2 \leq x < 3/4, \\ 3 & 3/4 \leq x < 1, \\ 0 & \text{otherwise} \end{cases}$$

in terms of its components in V_0, W_0 and W_1.