1. (10pts total; 5 pts each subitem) Suppose $K(x, y)$ is a continuous compactly supported function on $\mathbb{R} \times \mathbb{R}$. Define $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$T(f)(x) := \int_{y \in \mathbb{R}} f(y)K(x, y) \, dy.$$ 

Show that $T$ is a linear map and that its adjoint is given by

$$T^*(g)(x) := \int_{y \in \mathbb{R}} g(y)\overline{K(y, x)} \, dy.$$
2. (10pts total) Show that the sinc function

\[
sinc(x) := \begin{cases} 
\frac{\sin(\pi x)}{\pi x} & x \neq 0 \\
1 & x = 0
\end{cases}
\]

solves the refinement equation

\[
\phi(x) = \phi(2x) + \sum_{k \in \mathbb{Z}} \frac{2(-1)^k}{(2k + 1)\pi} \phi(2k - 2k - 1).
\]
3. (12pts total; 2pts for each subitem) Are the following filters (a) linear (b) time-invariant (c) causal?

1. \( (L_1 f)(t) = f(t) - \int_t^{t^2} f(x)dx, \)

2. \( (L_2 f)(t) := \int_{-\infty}^{\infty} f(x)e^{-(x-t)^2}dx. \)
4. (8pts total) Let \( \phi \) and \( \psi \) be the Haar scaling function and the Haar wavelet, respectively. We denote, as usual,

\[ V_j := \text{span}\{\phi(2^j \cdot -k), \ k \in \mathbb{Z}\}, \quad W_j := \text{span}\{\psi(2^j \cdot -k), \ k \in \mathbb{Z}\}. \]

Express the function \( f \) given by

\[
f(x) := \begin{cases} 
2 & 0 \leq x \leq 1/4, \\
-3 & 1/4 \leq x < 1/2, \\
1 & 1/2 \leq x < 3/4, \\
3 & 3/4 \leq x < 1, \\
0 & \text{otherwise}
\end{cases}
\]

in terms of its components in \( V_0 \), \( W_0 \) and \( W_1 \).