Math 104, homework # 8.

1. Prove or disprove the following statement: If a sequence \( \{x_n\} \) of real numbers satisfies the condition \( |x_n - x_{n+1}| < 1/2^n \), then \( \{x_n\} \) converges.

2. Give examples of an unbounded sequence that has no convergent subsequence and an unbounded sequence that has a convergent subsequence.

3. Let \( \{x_n\} \) be a sequence of real numbers such that \( \lim_{n \to \infty} x_{2n+1} = a, \lim_{n \to \infty} x_{2n} = b \), \( a \neq b \). Prove that \( \{x_n\} \) has exactly two limit points. How does this generalize? (State the generalization without a proof.)

4. Let \((X, d)\) be a complete metric space, let \( q \in (0, 1) \) be given, and let \( f : X \to X \) satisfy the condition
\[
d(f(x), f(y)) \leq q d(x, y) \quad \text{for all } x, y \in X.
\]
Show that the equation \( f(x) = x \) has exactly one solution. [Hint: take an arbitrary \( x_1 \) and build a sequence by the recurrence \( x_{n+1} := f(x_n) \).]