Math 104, homework # 8.

1. Prove or disprove the following statement: If a sequence $\{x_n\}$ of real numbers satisfies the condition $|x_n - x_{n+1}| < 1/2^n$, then $\{x_n\}$ converges.

2. Give examples of an unbounded sequence that has no convergent subsequence and an unbounded sequence that has a convergent subsequence.

3. Let $\{x_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} x_{2n+1} = a$, $\lim_{n\to\infty} x_{2n} = b$, $a \neq b$. Prove that $\{x_n\}$ has exactly two limit points. How does this generalize? (State the generalization without a proof.)

4. Let (X, d) be a complete metric space, let $q \in (0, 1)$ be given, and let $f : X \to X$ satisfy the condition

$$d(f(x), f(y)) \leq q\, d(x, y) \quad \text{for all } \ x, y \in X.$$

Show that the equation $f(x) = x$ has exactly one solution. [Hint: take an arbitrary x_1 and build a sequence by the recurrence $x_{n+1} := f(x_n)$.]