On properties of ω-automata using simulation distances

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Abstract

It is interesting to study properties of abstract metric spaces. Here we define metrics on the set of finite state automata $\mathcal{A}$ (over a fixed set of transition symbols Σ), and study their properties. We define simulation games on automata with a finite Σ, but we later move on to more general automata, where Σ is treated as a metric space itself. We then consider the original case and construct midpoints.

1 Introduction

Simulation distances between automata were defined in [2], using *quantitative simulation games*. The quantitative distance between two automata lies in the interval $[0, 1]$. The distances defined in [2] are directed. Using notation from that paper, we define the following notation:

$$d_{sim}(A, B) = d_{coverage}(A, B)$$

Where $A, B \in \mathcal{A}$ are finite state automata. We call the two player game corresponding to $d_{sim}(A, B)$ as A *simulating* B, where the maximising player makes moves on B, and the minimising player on A. Note that this is a directed distance. The objective function f that we consider is such that $f \in \{LimAvg, Disc_\lambda\}$.
We also define a bi-simulation distance $d_{bisim}(A, B)$, which is undirected, as follows. The maximising player can make a move on either A or B, and the minimising player has to reply by making a move on the other machine. It can be shown that $d_{bisim}(A, B)$ satisfies the triangular inequality$^{[proof needed]}$.

2 Simulation and bisimulation distance

2.1 Simulation

The game graph

Consider $A_1 = (Q_1, \Sigma, \delta_1, q_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, q_2) \in \mathcal{A}$. Let $\delta = d_{sim}(A_2, A_1)$, $m = |A_1|$, $n = |A_2|$. The game graph G for evaluating δ is defined in [2]. G is a bipartite graph, with edges between the two sets $Q_1 \times \{\#\} \times Q_2$ and $Q_1 \times \Sigma \times Q_2$. *wt* and *avgwt* are respectively the sum of weights, and the average weight of a set of edges of G.

Properties

- If A_2 exactly simulates A_1 (classically), then $d_{sim}(A_2, A_1) = 0$.
- $d_{sim}(A_2, A_1) = \underset{\sigma}{sup}\ \underset{\pi}{inf}\ f(\sigma, \pi) = \underset{\pi}{inf}\ \underset{\sigma}{sup}\ f(\sigma, \pi)$ where σ, π are **positional strategies** [3] of player 1,2
- d_{sim} is **directed** i.e. $d_{sim}(A_1, A_2)$ may not be equal to $d_{sim}(A_2, A_1)$
- d_{sim} satisfies **triangle inequality** i.e. $d_{sim}(A_1, A_3) \leq d_{sim}(A_1, A_2) + d_{sim}(A_2, A_3)$
- d_{sim} is a directed **hemimetric**.

We now look at two important lemmas concerning simulation.

Lemma 1. *$\delta = \frac{p}{q}$ where $gcd(p,q) = 1$ and $q \leq mn$.*

Proof. Positional strategies for each player will result in a run which will be a repeating cycle. By the fact that there exist positional strategies for the two players to achieve the value of the game [1, 3], δ, we can conclude that $\delta = \frac{wt(C)}{|C|}$ where C is the cycle in G resulting from the optimal positional strategies of both players. Since C is a cycle and G is a bipartite graph, with the smaller vertex set of size mn, $|C| \leq 2mn$ and $|C|$ is even. Also $wt(C)$ is even since the weight of each edge is 0 or 2. So $\delta = \frac{wt(C)}{|C|} = \frac{p}{q}$, $gcd(p,q) = 1$ and $q \leq mn$. □

Lemma 2. *Consider any cyclic walk W of length $2L$ in G (since the game graph is bipartite, length of any cyclic walk is even) then if $avgwt(W) < \delta$ then $avgwt(W) \leq \delta - \frac{1}{mnL}$*

Proof. $wt(W)$ is even. So let $wt(W) = 2\omega$.

$$avgwt(W) = \frac{wt(W)}{|W|} = \frac{\omega}{L} < \delta = \frac{p}{q}$$

$$\omega q < pL$$

Since ω, p, q, L are integers,

$$\omega q \leq pL - 1$$

$$avgwt(W) = \frac{\omega}{L} \leq \frac{p}{q} - \frac{1}{qL} \leq \delta - \frac{1}{mnL}$$

□

2.2 Bisimulation

The game graph

Consider $A_1 = (Q_1, \Sigma, \delta_1, q_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, q_2) \in \mathcal{A}$. Let $\delta = d_{bis}(A_2, A_1)$, $m = |A_1|$, $n = |A_2|$. The game graph $G^\dagger$ for evaluating δ is a slight variation of the simulation game graph. $G^\dagger$ has three vertex sets: $S_0 = Q_1 \times \{\#\} \times Q_2$, $S_1 = Q_1 \times \Sigma \times Q_2 \times \{1\}$ and $S_2 = Q_1 \times \Sigma \times Q_2 \times \{2\}$. The start state is $(q_1, \#, q_2)$. The player 1 vertices are S_0 and the player 2 vertices are $S_1 \cup S_2$. There are edges between S_0, S_1 and S_0, S_2. For $(u, \#, v) \in S_0$ and $(u', a, v, 2) \in S_2$, there is a weight 0 edge $(u, \#, v) \to (u', a, v, 2)$ iff A_1 contains the transition $u \xrightarrow{a} u'$. For all $(u', a, v, 2) \in S_2$, there is an edge e, $(u', a, v, 2) \to (u', \#, v')$ iff A_2 contains the transition $v \xrightarrow{b} v'$ for some $b \in \Sigma$. $wt(e) = 0$ iff $b = a$, 1 otherwise.
Between S_0 and S_1, edges are defined similarly, the difference being that S_0 to S_1 edges correspond to transitions in A_2, and the return edges to transitions in A_1.

Properties

- If A_2 exactly bisimulates A_1 (classically), then $d_{bis}(A_2, A_1) = 0$
- d_{bis} is **symmetric** i.e. $d_{bis}(A_1, A_2) = d_{bis}(A_2, A_1)$
- d_{bis} satisfies **triangle inequality**.
- d_{bis} is a **hemimetric**. It is a metric over equivalence classes of $\mathcal{A}_\Sigma$ where $A \equiv B$ iff $d_{bis}(A, B) = 0$
- $d_{bis}(A_1, A_2) \geq max(d_{sim}(A_1, A_2), d_{sim}(A_2, A_1))$

We note here that lemma 1 and 2 hold for bisimulation as well.

2.2.1 Bisimulation distance and equivalence classes

Define the relation $R \subset \mathcal{A} \times \mathcal{A}$ as $(A_1, A_2) \in R$ iff $d_{bis}(A_1, A_2) = 0$.

R is an equivalence relation

Reflexivity and symmetry are obviously true. Let us show that R is transitive as well. Take $A, B, C \in \mathcal{A}$, such that $(A, B) \in R$ and $(B, C) \in R$. Then by triangle inequality, $d_{bis}(A, C) = 0 \leq d_{bis}(A, B) + d_{bis}(B, C) = 0 + 0 = 0 \Rightarrow d_{bis}(A, C) = 0 \Rightarrow (A, C) \in R$.

It can also be shown that between any two automata in different equivalence classes, the value of the distance is independent of the automaton chosen to represent the class. Thus, d_{bis} is a metric on the set of equivalence classes. On $\mathcal{A}$ though, d_{bis} is a hemimetric.

3 Generalizing the alphabet Σ

Sometimes to understand the properties of some space we may have to generalise our domain. For example from $\mathbb{R}$ to $\mathbb{C}$. Until now, Σ was a finite set. We now generalize it to be a metric space itself. d_{sim}, d_{bis} can be defined over $\mathcal{A}_\Sigma$. The weight of an edge in the game graph now will be the distance between the labels on the transitions that players 1 and 2 take in a move. The basic question that we shall ask here is how do properties of Σ extend to those of $\mathcal{A}_\Sigma$.
As an example, $\Sigma = [0, 1] \cap \mathbb{Q}$ or $\Sigma = [0, 1] \subset \mathbb{R}$ can be used as an extension for $\Sigma' = \{0, 1\}$. $\Sigma = [0, 1]^n$ can be used as an extension for $\Sigma' = \{a_1, a_2, \cdots a_n\}$ where a_j is represented by $(\delta_{ij})_{i=1\cdots n}$ i.e. $a_1 = (1, 0, 0, \cdots, 0), a_2 = (0, 1, 0, \cdots, 0), \cdots, a_n = (0, 0, 0, \cdots, 1)$.

3.1 Extending functions from Σ to $\mathcal{A}_\Sigma$

We extend a function $\mu : \Sigma \times \Sigma \to \Sigma$ to $\tilde{\mu} : \mathcal{A}_\Sigma \times \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma$. This allows us to study properties of $\mathcal{A}_\Sigma$ depending on properties of Σ, and is useful in a lot of places as we shall see.

Let $A_1(Q_1, \Sigma, \delta_1, u_0)),\ A_2(Q_2, \Sigma, \delta_2, v_0)) \in \mathcal{A}_\Sigma$.
Let $\Pi : Q_1 \times \Sigma \times Q_2 \times \{1, 2\} \to (\Sigma \times Q_1) \cup (\Sigma \times Q_2)$ be the optimal positional strategy of player 2 in the A_2 *bisim* A_1 game. Let Π_Σ, Π_Q be the projections of Π.

Define $\tilde{\mu}(A_2, A_1) = A(A_1 \times A_2, \Sigma, \delta_A, (u_0, v_0))$ where $\forall(u, v) \in A_1 \times A_2$,
$(u, a, u') \in \delta_1 \Rightarrow ((u, v), \mu(a, \Pi_\Sigma(u', a, v, 2)), (u', \Pi_Q(u', a, v, 2)) \in \delta_A$ and
$(v, a, v') \in \delta_2 \Rightarrow ((u, v), \mu(a, \Pi_\Sigma(u, a, v', 1)), (\Pi_Q(u, a, v', 1), v') \in \delta_A$.

Similarly if simulation metric (A_2 *sim* A_1) is used, the optimal player 2 strategy $\Pi : Q_1 \times \Sigma \times Q_2 \times \to \Sigma \times Q_2$ and μ is extended as follows:
$\tilde{\mu}(A_2, A_1) = A(A_1 \times A_2, \Sigma, \delta_A, (u_0, v_0))$ where $\forall(u, v) \in A_1 \times A_2$,
$(u, a, u') \in \delta_1 \Rightarrow ((u, v), \mu(a, \Pi_\Sigma(u', a, v)), (u', \Pi_Q(u', a, v)) \in \delta_A$.

3.2 Connectedness

A topological space is a set X, with a topology τ defined on it. τ is a collection of subsets of X, satisfying the following properties.

1. $X \in \tau$, $\{\} \in \tau$.
2. τ is closed under arbitrary union.
3. τ is closed under finite intersection.

Sets in τ are called open sets.

Definition 1. *A topological space X with topology τ is* ***connected*** *if there do not exist disjoint non-empty open sets whose union is X.*

A topology is automatically defined when there is a metric d defined on X. For any $a \in X$, an open ball of radius δ around a, $B_\delta(a)$ is defined as $B_\delta(a) = \{x \in X | \ d(a, x) < \delta\}$. The open sets are then obtained using arbitrary union of open balls.

Theorem 1. *The metric space $\mathcal{A}$ of automata, when Σ is finite or countable, is not connected.*

Proof. Consider two cases.

Case 1 Σ is finite.

In case of finite Σ, distances are always rational. We construct disjoint non empty open sets $X, Y \subset \mathcal{A}$ whose union is $\mathcal{A}$. Choose an automaton $a \in \mathcal{A}$. a has just one state, and a single transition to itself on some symbol in $\sigma \in \Sigma$. Let $\epsilon = \frac{1}{\sqrt{2}}$, choose $X = B_\epsilon(a)$, and $Y = \mathcal{A} \setminus X$. Clearly, $X \cup Y = \mathcal{A}$. X is an open ball. Also, Y is open, since there is no automaton at distance ϵ (since it is irrational) from a. So, for any $b \in Y$, there is an open ball around b that is completely in Y. X is non empty, because $a \in X$. Also, there exists an automaton at distance 1 from a, which is obtained simply by changing the transition on σ to some $\overline{\sigma} \neq \sigma$. Thus Y is not empty. Hence $\mathcal{A}$ is not connected.

Case 2 Σ is countable.

This case is similar. Since Σ is countable, the number of automata are countable, hence the set of achievable values of distances is countable. Hence there exists an irrational value in the interval $[0, 1]$ which cannot be achieved (since irrationals are uncountable). Then we use the same construction as Case 1.

□

Definition 2. *A topological space X with topology τ is* **path connected** *if given any two elements $a, b \in X$, there exists a continuous function $f : [0, 1] \to X$ with $f(0) = a$ and $f(1) = b$.*

Theorem 2. *The metric space $\mathcal{A}$ of automata, is path connected when Σ is path connected.*

Proof. As shown in Section 2.2.1, d_{bis} is a metric on the equivalence classes induced by relation R. We show connectedness of this metric space. We shall prove that when Σ is path connected, $\mathcal{A}$ is path connected, hence connected. Consider two automata $A_1, A_2 \in \mathcal{A}$. We now construct a continuous path function $f : [0, 1] \to \mathcal{A}$ according to Definition 2.
Take any $\lambda \in [0, 1]$. Since Σ is path connected, for any $a_1, a_2 \in \Sigma$, we have a continuous function $f_{(a_1,a_2)} : [0, 1] \to \Sigma$, such that $f_{(a_1,a_2)}(0) = a_1$ and $f_{(a_1,a_2)}(1) = a_2$. Now define $\mu_\lambda : \Sigma \times \Sigma \to \Sigma$ as $\mu_\lambda(a, b) = f_{(a,b)}(\lambda)$. Then $f : [0, 1] \to \mathcal{A}$ is defined as $f(\lambda) = \tilde{\mu}_\lambda(A_1, A_2)$.
We now show that f satisfies all the desirable properties.

1. $f(0) = A_1$: When $\lambda = 0$, and we have labels a_1 and a_2, the transition in $f(0)$ is marked with $g(0) = a_1$. Thus all transitions on $f(0)$ correspond to transitions on A_1. Clearly $d_{bis}(f(0), A_1) = 0$ so $f(0) = A_1$ (upto equivalence classes).

2. $f(1) = A_2$: $f(1)$ will have transitions from A_2, and it is easy to see that $d_{bis}(f(1), A_2) = 0$.

3. f is continuous: Given $\epsilon > 0$, we show that there exists $\delta > 0$ such that $|\lambda_1 - \lambda_2| < \delta \Rightarrow d_{bis}(f(\lambda_1), f(\lambda_2)) < \epsilon$. Suppose the $\tilde{\mu}$ construction (the graph is isomorphic for any λ) has k edges. Then there are k pairs of symbols $a_i, b_i \in \Sigma$, and the continuous path function $f_{(a_i,b_i)}$ that connects them, for $i \in \{1, 2, \ldots, k\}$. Since each $f_{(a_i,b_i)}$ is continuous, there exists δ_i such that $|\lambda_1 - \lambda_2| < \delta_i \Rightarrow d_\Sigma(f_{(a_i,b_i)}(\lambda_1), f_{(a_i,b_i)}(\lambda_2)) < \epsilon$.
Choose $\delta = min(\delta_1, \delta_2, \ldots, \delta_k)$. Take $\lambda_1, \lambda_2 \in [0, 1]$ such that $|\lambda_1 - \lambda_2| < \delta$. Consider $f(\lambda_1)$ and $f(\lambda_1)$. We then suggest a strategy for the minimising player (player 2) that helps it attain a value less than ϵ, which would mean $d_{bis}(f(\lambda_1), f(\lambda_2)) < \epsilon$. Since $f(\lambda_1)$ and $f(\lambda_2)$ have the same state space, player 2 only follows the states taken by player 1. Since it is enough to look at positional strategies, the game will repeat as soon as both $f(\lambda_1)$ and $f(\lambda_2)$ reach the same pair of states. The error added up at each step is less than ϵ, and if the cycle length is c, then the value of the game is less than $\frac{c \cdot \epsilon}{c} = \epsilon$. This completes the proof.

□

3.3 Projection Theorem

Definition 3. *If (X, d) is a metric space and Y is a subspace of X, then the projection of $x \in X$ onto Y, $x|Y$, is defined as the set of closest points to x in Y.*

Projection may be empty sometimes. We say X is projectible on to Y if $x|Y \neq \phi \ \forall x \in X$.

Let $\Sigma' \subset \Sigma$ and assume Σ is projectible onto Σ'.

Let $A(Q, \Sigma, \delta_A, q_0) \in \mathcal{A}_\Sigma$. Define $A' \in \mathcal{A}_{\Sigma'}$ as follows:
$A'(Q, \Sigma', \delta_{A'}, q_0)$ where $(q, a, q') \in \delta_A \Rightarrow (q, a', q') \in \delta_{A'}$ where $a' \in a|\Sigma'$.

Theorem 3. *$\mathcal{A}_\Sigma$ is projectible onto $\mathcal{A}_{\Sigma'}$ and $A' \in A|\mathcal{A}_{\Sigma'}$*

Proof. Let $B' \in \mathcal{A}_{\Sigma'}$. Let $d_{bis}(A', A) = f(\sigma^*, \pi^*)$ where σ^* and π^* are optimal strategies of player 1 and 2 respectively. σ^* and π^* together form a cycle in the game graph which corresponds to a walk W in A. Let $a(e)$ represent the label of edge e.

$$d(A', A) = \sum_{e \in W} a(e)|\Sigma'$$

If in the A' *bisim* A game, if player 1 follows the strategy σ_W, of following the walk W again and again, and if the player 2 follows his optimal strategy ν^* then

$$d(A', A) = \sum_{e \in W} a(e)|\Sigma' \geq f(\sigma_W, \nu^*) \geq d(B', A)$$

□

Note that although we stated the projection theorem for symmetric metrics it holds for directed metrics too.

3.4 Denseness

Definition 4. *A subset A of a topological space X is dense in X if for any point $x \in X$, any neighborhood of x contains at least one point from A.*

Suppose we have two sets of alphabet Σ', Σ such that $\Sigma' \subset \Sigma$. We prove the following theorem.

Theorem 4. *$\mathcal{A}_{\Sigma'}$ is dense in $\mathcal{A}_\Sigma$ iff Σ' is dense in Σ*

Proof. We prove this for bisimulation. The proof for simulation is similar. Suppose Σ' is dense in Σ. Then given $A \in \mathcal{A}_\Sigma$ and $\epsilon > 0$, we construct $A' \in \mathcal{A}_{\Sigma'}$ such that $d_{bis}(A, A') < \epsilon$. To do this, we simply, replace the symbol a_e on each transition e of A by a symbol $a'_e \in \Sigma'$ such that $d_\Sigma(a_e, a'_e) < \epsilon$. Then player 2 only follows the states that player 1 takes, and this strategy ensures a distance less than ϵ, so $d_{bis}(A, A') < \epsilon$.
Now suppose $\mathcal{A}_{\Sigma'}$ is dense in $\mathcal{A}_\Sigma$. Given $a \in \Sigma$ and $\epsilon > 0$, consider $A \in \mathcal{A}_\Sigma$, where A has a single state with a self loop on symbol a. Since $\mathcal{A}_{\Sigma'}$ is dense in $\mathcal{A}_\Sigma$, there exists $A' \in \mathcal{A}_{\Sigma'}$ such that $d_{bis}(A, A') < \epsilon$. Suppose the A *bisim* A' game runs for c steps before repeating (for optimal strategies). Then we have $\frac{d_\Sigma(a,a_1)+d_\Sigma(a,a_2)+\cdots+d_\Sigma(a,a_c)}{c} < \epsilon$ where $a_1, a_2, \cdots, a_n$ are transitions taken by player 2. For this to be true, we must have $d_\Sigma(a, a_i) < \epsilon$ for some i. This completes the proof. □

As an example, $\mathcal{A}_\mathbb{Q}$ is dense in $\mathcal{A}_\mathbb{R}$ because $\mathbb{Q}$ is dense in $\mathbb{R}$. But $\mathcal{A}_{\{0,1\}}$ is not dense in $\mathcal{A}_{[0,1]}$.

3.5 Incompletess of $\mathcal{A}$

The metric space of automata $\mathcal{A}$ is over Σ, where distance is the bisimulation distance. If Σ is a metric space, then for any $a, b \in \Sigma$, the midpoint of a and b exists in Σ.

Definition 5. *Given a metric space $\mathcal{T}$, with distance function $d_\mathcal{T} : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$, an infinite sequence $(x_1, x_2, x_3, \ldots)$ in $\mathcal{T}$ is called **Cauchy**, if for any $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall m, n > n_0, d_\mathcal{T}(x_m, x_n) < \epsilon$.*

Definition 6. *A metric space $\mathcal{T}$ is called **complete** if every Cauchy sequence in $\mathcal{T}$ converges.*

We now construct sequences of cyclic automata with labels from the set $\{0, 1\}$. Then a cyclic automaton can be represented by a binary string.
Given a binary string $\alpha = b_1 b_2 \ldots b_i \ldots b_n$, define $fl_i(\alpha) = b_1 b_2 \ldots \overline{b_i} \ldots b_n$.
Let $a_0 = 0$. Given a_k, we choose $i \leq |a_k|$. Then $a_{k+1} = (a_k) \cdot (fl_i(a_k))$, where the $\cdot$ is used to append two strings. In this way, the sequence $\{a_i\}$ that gets constructed depends solely on the choice of the position of the bit that we choose to flip. For example, $a_0 = 0$, $a_1 = 01$, $a_2 = 0100$, $a_3 = 01000110$

and so on. At any step, choosing a different bit to flip would result in a different sequence from that position onwards. Like when we have $a_2 = 0100$ as above, if instead of the third bit from left, we choose the first bit from left, we have $a_3 = 01001100$.

Lemma 3. *Any sequence $\{a_i\}$ of the form defined above is a Cauchy sequence.*

Proof. It is easy to see that $d_{bis}(a_i, a_{i+1}) = \frac{1}{2^{i+1}}$. Thus, for $j > i$, using the triangle inequality, we have $d_{bis}(a_i, a_j) \leq \frac{1}{2^{i+1}} + \frac{1}{2^{i+2}} + \ldots + \frac{1}{2^j} = \frac{1}{2^i} - \frac{1}{2^j}$. For arbitrary $\epsilon > 0$, we choose n_0 such that $\frac{1}{2^{n_0}} < \epsilon$. Clearly then, $\forall m, n \cdot m > n > n_0$, $d_{bisim}(a_m, a_n) \leq \frac{1}{2^n} - \frac{1}{2^m} < \frac{1}{2^{n_0}} < \epsilon$. ☐

Lemma 4. *Consider two different sequences $\{a_i\}$ and $\{b_i\}$. Let k be an index such that $b_k \neq a_k$. Suppose b_k and a_k mismatch at exactly η postitions. Then, b_{k+1} and a_{k+1} have to mismatch at atleast $2\eta - 2$ positions.*

Proof. The first halfs of b_{k+1} and a_{k+1} have η mismatches. The second halfs have to have atleast $\eta - 2$, because atmost 2 mismatches can be corrected by flipping 1 bit each in b_k and a_k. ☐

Lemma 5. *Two sequences $\{a_i\}$ and $\{b_i\}$ with more than 2 differences at some index k cannot converge to the same limit.*

Proof. Suppose $\{a_k\}$ and $\{b_k\}$ have $\eta > 2$ mismatches. Then $d_{bis}(a_k, b_k) = \frac{\eta}{2^k}$. So we have at least $2\eta - 2$ mismatches between $\{a_{k+1}\}$ and $\{b_{k+1}\}$, at least $4\eta - 6$ mismatches between $\{a_{k+2}\}$ and $\{b_{k+2}\}$, and so on (using lemma 4). This way in general, we have atleast $2^m\eta - 2^{m+1} + 2$ mismatches between $\{a_{k+m}\}$ and $\{b_{k+m}\}$. Hence, we have $d_{bis}(a_{k+m}, b_{k+m}) \geq \frac{2^m\eta - 2^{m+1} + 2}{2^{k+m}} = \frac{\eta - 2}{2^k} + \frac{2}{2^{k+m}} > \frac{\eta - 2}{2^k}$, which is a positive constant since $\eta > 2$. This holds for all values of m. Suppose $\{a_i\}$ and $\{b_i\}$ have the same limit A. Then we choose $\epsilon > 0$ such that $\frac{\epsilon}{2} < \frac{\eta - 2}{2^k}$. Then, there exists a common threshold n_0 (take the larger one of the thresholds for the two sequences) such that $\forall n > n_0$, $d_{bis}(a_n, A) < \frac{\epsilon}{2}$ and $d_{bis}(b_n, A) < \frac{\epsilon}{2}$. By the triangle inequality, $d_{bis}(a_n, b_n) \leq d_{bis}(a_n, A) + d_{bis}(b_n, A) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This is a contradiction. ☐

Lemma 6. *The set $\mathcal{S}$ of sequences, such that every pair of sequences in $\mathcal{S}$ converges to a different limit is uncountable.*

Proof. Start with $a_0 = 0000$ and $b_0 = 1111$. Then any two sequences, one generated from a_0 and the other from b_0 cannot converge to the same limit by lemma 5. Now, consider a_0, take two different branches from a_0. Say $a_1 = 00000001$ and $a'_1 = 00000010$. Suppose now that we flip the same bit, so that a_2 and a'_2 have 4 mismatches. After that, we are at the same situation as initially with a_0 and b_0. This way, we can construct an infinite depth binary tree, with each path resulting in a sequence such that no two limits can be same. Clearly, the set of these sequences $\mathcal{B} \subset \mathcal{S}$. Since we have a binary tree, we have that $|\mathcal{B}| = 2^{\mathbb{N}}$. Thus, $\mathcal{S}$ is uncountable. ☐

Theorem 5. *$\mathcal{A}$ is incomplete.*

Proof. The proof is by contradiction. Assume that every Cauchy sequence in $\mathcal{A}$ converges.

Case 1 Σ is finite.
By lemma 6, the number of different sequences, all with different limits, is uncountable. But the set of finite automata on Σ is countable. This is a contradiction.

Case 2 Σ is an arbitrary metric space, $\{0, 1\} \subset \Sigma$.
By Theorem 3, it is not possible for an automaton over $\{0, 1\}$ to come arbitrarily close to an automaton in Σ. Hence the limit cannot be a general automaton over Σ. But by Case 1, there is no limit over $\{0, 1\}$. This is a contradiction, which completes the proof.

☐

4 Midpoints

4.1 Midpoint when Σ has midpoints

4.1.1 Simulation

Let $A_1, A_2 \in \mathcal{A}_\Sigma$ with $d_{sim}(A_2, A_1) = \delta$.
Let $A = \tilde{\mu}(A_2, A_1)$ where $\mu : \Sigma \times \Sigma \to \Sigma$ is the midpoint map for Σ

Theorem 6. *$d_{sim}(A, A_1) = d_{sim}(A_2, A) = \frac{\delta}{2}$ i.e. $A = midpoint(A_2, A_1)$ i.e. $\tilde{\mu}$ is the midpoint map for $\mathcal{A}_\Sigma$.*

Proof. We will first show $d_{sim}(A, A_1) \leq \frac{\delta}{2}$ and then $d_{sim}(A_2, A) \leq \frac{\delta}{2}$. Since we have $d_{sim}(A_2, A) + d_{sim}(A, A_1) \geq \delta$ because of triangle inequality, we get the required result. Consider A *sim* A_1 game. Let σ be any strategy of player 1. Let μ be the strategy of player 2 which is to match the first coordinate of his current state with player 1 current state. Clearly

$$d_{sim}(A, A_1) \leq f(\sigma, \mu) \leq \frac{\delta}{2}$$

Now consider the A_2 *sim* A game.Let ν be any strategy of player 1. Let π be the strategy of player 2 which is basically playing strategy Π depending on the first coordinate of player 1 current state. Again

$$d_{sim}(A_2, A) \leq f(\nu, \pi) \leq \frac{\delta}{2}$$

□

4.1.2 Bisimulation

Let $A_1, A_2 \in \mathcal{A}_\Sigma$ with $d_{bis}(A_2, A_1) = \delta$.
Let $A = \tilde{\mu}(A_2, A_1)$ where $\mu : \Sigma \times \Sigma \to \Sigma$ is the midpoint map for Σ

Theorem 7. *$d_{bis}(A, A_1) = d_{bis}(A_2, A) = \frac{\delta}{2}$ i.e. $A = midpoint(A_2, A_1)$ i.e. $\tilde{\mu}$ is the midpoint map for $\mathcal{A}_\Sigma$.*

The proof is similar to that of simulation case.

Focusing on finite Σ

In this case, Σ does not have midpoints. Clearly the constuction in Section 4.1 wouldn't work. We have to try and modify it to suit this case.
Consider $A_1, A_2 \in \mathcal{A}$. Suppose $d_{sim}(A_2, A_1) = \delta$. We look at the following definition:

$$A = \underset{\overline{A} \in \mathcal{A}}{\operatorname{argmin}}(\langle A_2, \overline{A}, A_1 \rangle)$$

where the proximity $\langle A_2, \overline{A}, A_1 \rangle = max(d(\overline{A}, A_1), d(A_2, \overline{A}))$ for $d \in \{d_{sim}, d_{bis}\}$. Note that by the triangle inequality, $d(\overline{A}, A_1) + d(A_2, \overline{A}) \geq d(A_2, A_1)$. Hence it is not possible that both $d(\overline{A}, A_1) < \frac{\delta}{2}$ and $d(A_2, \overline{A}) < \frac{\delta}{2}$ hold simultaneously. This implies that $\langle A_2, \overline{A}, A_1 \rangle$ is bounded below by $\frac{\delta}{2}$. This bound is tight, because there exist examples which attain this bound. The lower bound is achieved if A is the midpoint.

We also define middle points as those $A \in \mathcal{A}$ for which $\langle A_2, A, A_1 \rangle < d(A_2, A_1)$. It is not known whether midpoints always exist. We shall first look at the problem of finding middle points. Let the set of all middle points be denoted by $\mathcal{M}(A_2, A_1)$.

4.2 The case of $Disc_\lambda$

$Disc_\lambda$ stands for the discounted distance objective, with paramenter λ. It is easy to show that given $A_1, A_2 \in \mathcal{A}$, $\mathcal{M}(A_2, A_1)$ may be empty. Let $A_1 = (Q_1, \Sigma, \delta_1, q_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, q_2)$. $\Sigma = \{a, b\}$. $Q_1 = \{q_1, q_1'\}$. $\delta_1(q_1, a) = q_1'$, $\delta_1(q_1', a) = q_1'$ and $\delta_1(q_1, b) = q_1'$. Similarly, $Q_2 = \{q_2, q_2'\}$. $\delta_2(q_2, b) = q_2'$, $\delta_2(q_2', a) = q_2'$ and $\delta_2(q_2, b) = q_2'$.

Clearly, $d_{sim}(A_2, A_1) = 1 - \lambda$. Suppose $\mathcal{M}(A_2, A_1)$ is non empty. Then there exists $A \in \mathcal{M}(A_2, A_1)$. The start state of A must have a transition on the symbol b, otherwise $d_{sim}(A, A_1) \geq 1 - \lambda$. But if that happens, then $d_{sim}(A_2, A) \geq 1 - \lambda$, because the maximising player will take the b transition at the start in A, which the opponent cannot match in A_2.

So from now on, we shall only deal with the *Limavg* objective.

Theorem 8. *For $A_1, A_2 \in \mathcal{A}$, when the LimAvg objective is used, $\mathcal{M}(A_2, A_1)$ is non-empty.*

We shall prove it in Section 4.4.

4.3 Preliminaries

4.3.1 Tree unrolling of finite state machine A

Descriptive definition

Let $A = (Q, \Sigma, \delta, q_0)$, where symbols have their usual meaning. A may be non-deterministic (but without ϵ transitions). The tree unrolling $TU(A)$ is defined as the finite state machine $(2^Q \times Q, \Sigma, \delta', q_0') \in \mathcal{A}$. The start state $q_0' = (\{\}, q_0)$. $q' \in \delta(q, a)$ and $q \notin S \Rightarrow (S \cup \{q\}, q') \in \delta'((S, q), a)$. If $q \in S$, then $\delta'((S, q), a) = \{\}$. By according to this structure, the graph of $TU(A)$ is a tree. The states (S, q) where $q \in S$ are the leaves. Tree unrolling is based on finite mean payoff games [1].

Complexity of tree unrolling

We note here that the worst case size complexity of the tree unrolling is exponential. Suppose the maximum outdegree in A is d. Also, the depth of the unrolling before repetition of a label could be at most n. Thus the number of states in $TU(A)$ is $\mathcal{O}(d^n)$.
As an example, suppose the graph of A is the complete graph on n vertices K_n, then the number of states in $TU(A)$ is $(n-1) + (n-1)(n-2) + \ldots + (n-1)!$ (the i^{th} level has $n - i - 1$ live nodes as children). This equals $\binom{n-1}{1} + 2!\binom{n-1}{2} + \ldots + (n-1)!\binom{n-1}{n-1} \in \Omega(2^n)$, and is hence exponential.

There are examples for A where the size of $TU(A)$ is exponential, even when A has constant tree width or constant tree congestion.

4.3.2 Serial combination of unrolled trees

Given $B, C \in \mathcal{A}$, with $B = (Q, \Sigma, \delta_B, q_1)$ and $C = (Q, \Sigma, \delta_C, q_1)$, with states $\{q_1, q_2, \ldots, q_n\}$, we construct $B_1, B_2, \ldots, B_n, C_1, C_2, \ldots, C_n \in \mathcal{A}$. For $i \in \{1, 2, \ldots, n\}$, define $B_i = TU((Q, \Sigma, \delta_B, q_i))$ and $C_i = TU((Q, \Sigma, \delta_C, q_i))$. We use the notation $\bigsqcup(B)$ to denote the n disconnected graphs $B_1, B_2, \ldots, B_n$.

Once we create B_i and C_i for all i, we start connecting them as follows.
Consider a dead state q_k of B_i. Suppose its parent is q_k'. Then B_i has the transition $q_k' \xrightarrow{a} q_k$. We delete q_k, add a transition from q_k' to the start state of C_k on the symbol a. We do this for all dead states for every B_i. Similarly, we delete dead states for every C_i, and add transitions back to the corresponding B_j. This construction is denoted by $B \otimes C$. States of $B \otimes C$ are labelled as tuples from $Q \times \{1, 2, \ldots, n\} \times \{0, 1\}$. For a state q of B_i, its label is the 3-tuple $(m, i, 0)$, where m was its label in $TU(B)$. For a state q of C_i, its label is $(m, i, 1)$, where m was its label in $TU(C)$.
We shall use this *serial combination* construction in Section 4.4.

Lemma 7. *Given $B, C \in \mathcal{A}$, no simple cycle Γ of $B \otimes C$ can have two states with the same label (where a label belongs to the set $Q \times \{1, 2, \ldots, n\} \times \{0, 1\}$). Hence $|\Gamma| \leq 2n^2$.*

Proof. We prove the lemma by contradiction. Two nodes have the same label iff they belong to the same B_i or the same C_i. Without loss of generality, we assume that two states q and q' belonging to a simple cycle Γ of $B \otimes C$ having the same label, belong to B_i. If we start traversing the cycle starting at q, then to reach q', the only way is to reach the start state q_i of B_i. The cycle then follows on to q', but after which, the only way to reach q is to reach the start state q_i of B_i again. But then we have already traversed a simple cycle, without reaching q where we started. This is a contradiction. Since the number of different labels is at most $n \cdot n \cdot 2 = 2n^2$. This implies $|\Gamma| \leq 2n^2$. □

4.4 Proof of theorem 8

Construction

Let $A_1(Q_1, \Sigma, \delta_1, u_0))$, $A_2(Q_2, \Sigma, \delta_2, v_0))$ be two automata with $d_{sim}(A_2, A_1) = \delta$.
Let $C = \tilde{\mu}(A_2, A_1)$ where $\mu(a, b) = a$ and $D = \tilde{\nu}(A_2, A_1)$ where $\nu(a, b) = b$
Note that the graphs of C and D are isomorphic and just transition labels are different.

Let $T_C = \bigsqcup(C)$ and $T_D = \bigsqcup(D)$. Let $A = C \otimes D$.

Theorem 9. *$d_{sim}(A_2, A) \leq \delta - \frac{1}{2(mn)^3}$, $d_{sim}(A, A_1) \leq \delta - \frac{1}{2(mn)^3}$ where $m = |Q_1|$, $n = |Q_2|$*

Proof. We will first show $d_{sim}(A, A_1) \leq \delta - \frac{1}{2(mn)^3}$. The proof for $d_{sim}(A_2, A) \leq \delta - \frac{1}{2(mn)^3}$ is similar. Consider the A sim A_1 game. Since optimal positional stategies exist for players 1 and 2 which results in the value of the game, we will look at only positional stategies.
Let σ_1 be any positional strategy of player 1 playing on A_1. If the player 2 on A follows the positional strategy π_1 which is to match the first coordinate of the label on his current state with the current state of player 1 on A_1 then we will show that the outcome of the game on (σ_1, π_1) , $\Psi(\sigma_1, \pi_1) \leq \delta - \frac{1}{2(mn)^3}$. Since $d_{sim}(A, A_1) \leq \max_{\sigma_1} \Psi(\sigma_1, \pi_1)$ we get the required result.

Since σ_1, π_1 are positional strategies the game will result in a cycle Γ' in the game graph which is ultimately repeated. So it is enough to show that average weight $avgwt(\Gamma') \leq \delta - \frac{1}{2(mn)^3}$. The cycle Γ' in general corresponds to a walk Γ in A.

Observe that throughout the game the first coordinate of the label on player 2 state is same as the player 1 state. So a state repeating in Γ implies that player 1 is in the same state as he was earlier, which contradicts the fact that Γ' is a cycle. So Γ is a cycle. By lemma 7, $|\Gamma| \leq 2(mn)^2$.

Define the graph $G(Q_1 \times Q_2, E) : (u, v) \rightarrow (u', v') \in E \Leftrightarrow u \rightarrow u'$ and $v \rightarrow v'$ in A_1 and A_2 respectively.
Consider the sequence of labels in Γ in G. They form a cyclic walk W as shown in Figure 1. The solid lines correspond to the run in T_C and the dashed lines to that of T_D.

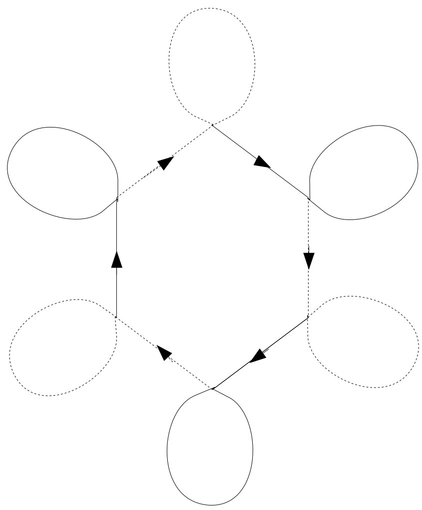

Figure 1: Cyclic walk W

Each edge in G corresponds to two moves in the game graph of A_2 sim A_1. Let $e = (u, v) \xrightarrow{a} (u', v')$ be an edge in W.
If e is in T_C then it corresponds to the moves $(u, \#, v) \xrightarrow{a} (u', a, v) \xrightarrow{*} (u', \#, v')$ where $* = \Pi_\Sigma(u', a, v)$. If e is in T_D then it corresponds to the moves $(u, \#, v) \xrightarrow{*} (u', *, v) \xrightarrow{a} (u', \#, v')$ where $a = \Pi_\Sigma(u', *, v)$

Define weight, $wt(e) = \begin{cases} 0 & \text{if } a = * \\ 1 & \text{if } a \neq * \end{cases}$ and weight of a set of edges is a sum of the weights of its edges.

Let W_{solid}, W_{dotted} be the parts of W corresponding to runs in T_C and T_D respectively. Let $L = |W| = |C| \leq 2(mn)^2$

$$avgwt(C') = \frac{wt(W_{dotted})}{L} \quad avgwt(W) = \frac{wt(W_{dotted}) + wt(W_{solid})}{L}$$

This walk W corresponds to a walk in the game graph of A_2 *sim* A_1 game which results when player 1 on A_1 follows σ_1 and player 2 on A_2 follow Π. So $avgwt(W) \leq \delta$.

$$avgwt(C') = \frac{wt(W_{dotted})}{L} \leq \frac{wt(W_{dotted}) + wt(W_{solid})}{L} = avgwt(W) \leq \delta$$

Case i : $avgwt(W) < \delta$
Then by lemma 2

$$avgwt(C') \leq avgwt(W) \leq \delta - \frac{1}{mnL} \leq \delta - \frac{1}{2(mn)^3}$$

Case ii : $avgwt(W) = \delta$
For every walk K' in G that corresponds to a run where player 2 plays strategy Π, $avgwt(K') \leq \delta$. Since W is such a walk, any cycle K in W has $avgwt(K) \leq \delta$. Therefore, $avgwt(W) = \delta \implies avgwt(K) = \delta$ for every cycle K in W. Since there exists at least one solid cycle K_{solid} in W (see Figure 1) with $avgwt(K_{solid}) = \delta > 0$, $wt(W_{solid}) \geq avgwt(K_{solid}) \cdot |K_{solid}| \geq 1$.

$$\delta = avgwt(W) = \frac{wt(W_{dotted}) + wt(W_{solid})}{L} \geq avgwt(C') + \frac{1}{L}$$

$$avgwt(C') \leq \delta - \frac{1}{2(mn)^2} \leq \delta - \frac{1}{2(mn)^3}$$

□

Clearly the constructed automaton $A \in \mathcal{M}(A_2, A_1)$. This proves Theorem 8.

4.5 Optimising the construction of $A \in \mathcal{M}(A_2, A_1)$

In Section 4.4, we constructed $C \otimes D \in \mathcal{A}$. Instead of constructing the serial combination of T_C and T_D directly, we do the following optimisation. For every dead state q of T_C, we assign a value. Since T_C contains disconnected graphs C_i, suppose $q \in C_i$ for some i. We consider the labels of the path from the root state of C_i to state q. We then consider the walk formed by the set of these labels in the graph $G(A_1 \times A_2, E)$ (defined in Section 4.4). Since q is a dead state, this walk contains exactly one cycle. The average weight of this cycle is the value of q.
Now, $\delta = d_{sim}(A_2, A_1)$. For every dead state q of T_C, suppose q' is its parent, we deleted q and added a transition from q' to the root state of some component D_j of T_D. We now do this only when the value of q greater than $\frac{\delta}{2}$. Otherwise, we delete q, and add a transition from q' to C_j instead of D_j. Essentially, if the value of the cycle covered in the T_C part is not more than $\frac{\delta}{2}$, then taking that cycle does not allow moving out to T_D. Dead states of T_D are deleted similarly.
Suppose this new construction is denoted by A'. Then

$$\langle A_2, A', A_1 \rangle \leq \langle A_2, A, A_1 \rangle$$

So, A' is in general a better approximation to the mid point (if it exists) than A. In the worst case, the two values above could be equal, although heuristically this does better. In [2], we consider the examples *TwoBs* and *OneBs*. Taking $A_1 = TwoBs$ and $A_2 = OneBs$, we have $d_{sim}(A_2, A_1) = \frac{1}{3}$. The old construction gives $\langle A_2, A, A_1 \rangle = \frac{1}{4}$, and the new optimised construction gives $\langle A_2, A', A_1 \rangle = \frac{1}{6}$, which is in fact, the actual midpoint.

4.6 Middle point construction for bisimulation

Let $A_1, A_2 \in \mathcal{A}_\Sigma$ with $d_{bis}(A_2, A_1) = \delta$.
Let $C = \tilde{\mu}(A_2, A_1)$ where $\mu(a,b) = a$ and $D = \tilde{\nu}(A_2, A_1)$ where $\nu(a,b) = b$
Let $T_C = \bigsqcup(C)$ and $T_D = \bigsqcup(D)$. Let $A = C \otimes D$.

Theorem 10. *$\tilde{\mu}$ is a midppoint map for $\mathcal{A}_\Sigma$. $d_{bis}(A_2, A) \leq \delta - \frac{1}{2(mn)^3}$, $d_{bis}(A, A_1) \leq \delta - \frac{1}{2(mn)^3}$ where $m = |A_1|$, $n = |A_2|$*

The proof of this theorem is very similar to that of directed case so we omit it here.

References

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