Comparing Expressive Power of Timed Logics
BTP report

Nishant Totla

Advisors
Prof. S. Krishna
Prof. Paritosh. K. Pandya

June 26, 2012

Abstract

Timed temporal logics encompass a huge diversity of operators, as a result of which, the decidability and expressiveness properties vary considerably. Both pointwise and continuous interpretations of temporal logics can be defined (over finite words), and properties vary based on the interpretation too. The Ehrenfeucht-Fraïssé games (EF games) for Linear Temporal Logic, as defined by Etessami-Wilke [3] are extended to EF games for Metric Temporal Logic (MTL) [6], for both the pointwise and continuous interpretations. Using the corresponding EF Theorem, we show various results comparing the expressive power of various fragments of the full MTL.

In particular, we also establish a hierarchy for MTL, based on the largest constant that can appear in the interval constraints. We show using EF games that this hierarchy holds for pointwise timed MTL, but collapses in the continuous case.

For the continuous case, the major goal was to attack the open problems of comparing MTL with timed past and untimed past. The question is whether $MTL^c[\mathbf{U}_I, \mathbf{S}_I]$ is strictly stronger than $MTL^c[\mathbf{U}_I, \mathbf{S}]$. We have been able to solve this for unary modalities.

Contents

Chapter 1

Studying logics

Logics are well defined languages that can be used to study properties of various kinds of structures. Like any other language, a logic is made up on syntax and semantics. Syntax denotes the structure of the formulas that are part of the logic. Syntax is usually depicted as a recursive context-free grammar, and arbitrary formulas can be created that fit this syntax. Formulas by themselves carry no meaning. Semantics are rules by which meaning is assigned to every syntactically correct formula of the logic.

Formally, given a logic $\mathcal{L}$, and a formula $\phi \in \mathcal{L}$, we define a set of allowed models $\mathcal{M}$ to define the semantics for $\mathcal{L}$. A model $A \in \mathcal{M}$ is a formal structure which is used to interpret the formula ϕ. For every formula and every model, we define rules to interpret the formula on the model. In a well-defined logic, the model either satisfies the formula (denoted $A \models \phi$) or does not satisfy (denoted $A \not\models \phi$).

For instance, the set of models could be all graphs, and the logic might be used to qualify properties of graphs - such as connectedness, planarity or being bipartite etc. A logic may or may not be strong enough to qualify such properties, and it is a worthwhile question to ask whether the logic one is working with is "strong" enough or not.

1.1 Why is studying logics important?

Logics are very powerful. Expressing complex properties can be done very compactly with logics. This is where they are very useful - compact expression of specifications. On the other hand, being too powerful might make the satisfiability checking problem for that logic undecidable, so it makes sense to deal with as weak a syntax as will suit our purpose. And this is precisely why it is important to study logics, and the properties they can express. Logics are also deeply connected to other areas of theory like complexity and probability. So answering questions about logics might help in answering questions in those areas.

Chapter 2

Introduction to temporal logics

Temporal logic is a language for expressing relationships between the order of events occuring over time. It is used to describe any system of rules and symbolism for representing, and reasoning about, propositions qualified in terms of time. Temporal logic has found an important application in formal verification, where it is used to state requirements of hardware or software systems. For instance, one may wish to say that whenever a request is made, access to a resource is eventually granted, but it is never granted to two requestors simultaneously. Such a statement can conveniently be expressed in a temporal logic.

We shall consider temporal logics that express properties of events on a linear time line. We then employ logical and modal operators to express properties of such events in relation to each other. We shall now look at some examples of untimed and timed temporal logics.

2.1 Preliminaries

We shall first define timed words as models over which temporal logic formulas are interpreted. Let $AP = \Sigma$ be the set of atomic propositions. These are events that occur at various time instances. A finite timed word is a finite sequence $\rho = (\sigma_1, \tau_1), (\sigma_2, \tau_2), \cdots, (\sigma_n, \tau_n)$ where $\sigma_i \in \Sigma$, $\tau_i \in \mathbb{R}$, such that the sequence of time stamps is non-decreasing: $\forall i < n \cdot \tau_i \leq \tau_{i+1}$. This gives weakly monotonic timed words. If the time stamps are strictly increasing, then we have strictly monotonic timed words: $\forall i < n \cdot \tau_i < \tau_{i+1}$. The length of ρ is denoted by $\#\rho$ and $dom(\rho) = \{1, 2, \cdots, \#\rho\}$. For convenience, we assume $\tau_1 = 0$. The timed word can alternatively be represented as $\rho = (\overline{\sigma}, \overline{\tau})$, with $\overline{\sigma} = \sigma_1, \sigma_2, \cdots, \sigma_n$ and $\overline{\tau} = \tau_1, \tau_2, \cdots, \tau_n$. Let $untime(\sigma) = \overline{\sigma}$.
What we defined above are timed words for a pointwise interpretation. We can similarly define continuous timed words. Note that in pointwise words, events happen at specified times (τ_1, τ_2 etc.), and nothing happens "in between" these time instances. In a continuous word, each position t on the time line is a valid point, and we can ask whether or not a certain proposition of Σ is true at that time[1] t. We consider only bounded timed words. Let $time(\rho)$ denote the largest time point till which the word extends.

2.2 Linear Temporal Logic (LTL)

LTL uses the usual logical operators ($\wedge, \vee, \neg$) and modal operators (**U**,**S**,**O**,**F**,**P** etc.). We define LTL with Until (**U**) and Since (**S**) modalities. The others, whenever defined, can be expressed using these.

2.2.1 Syntax

$$\phi ::= a \mid \phi \wedge \phi \mid \neg\phi \mid \phi \mathbf{U} \phi \mid \phi \mathbf{S} \phi$$

where a is an atomic proposition, ie $a \in \Sigma$.

[1] Just as a clarification note, we can assume that exactly one atomic proposition holds true at a time instance. But this is not restrictive, because even if multiple propositions (say σ_1, σ_2) hold at time t we can always construct a new set of atomic propositions with the new proposition d_{12} denoting $\sigma_1 \wedge \sigma_2$.

2.2.2 Semantics

Let ρ be a pointwise timed word and $i \in dom(\rho)$. We define sematics for when the word ρ models formula ϕ at point i as follows:

$$\begin{array}{rcl} \rho, i \models a & \text{iff} & \sigma_i = a \\ \rho, i \models \neg\phi & \text{iff} & \rho, i \not\models \phi \\ \rho, i \models \phi_1 \wedge \phi_2 & \text{iff} & \rho, i \models \phi_1 \text{ and } \rho, i \models \phi_2 \\ \rho, i \models \phi_1 \mathbf{U} \phi_2 & \text{iff} & \exists j > i \cdot \rho, j \models \phi_2 \text{ and } \forall i < k < j \cdot \rho, k \models \phi_1 \\ \rho, i \models \phi_1 \mathbf{S} \phi_2 & \text{iff} & \exists j < i \cdot \rho, j \models \phi_2 \text{ and } \forall j < k < i \cdot \rho, k \models \phi_1 \end{array}$$

We say that the word ρ models formula ϕ (denoted $\rho \models \phi$) iff $\rho, 1 \models \phi$. This is known as the anchored interpretation. The semantics for a continuous timed word ϕ are similar, but defined at a time point t instead. Define $\rho(t)$ to be the atomic proposition that holds true at time instance t. In this case, we say ρ models ϕ iff $\rho, 0 \models \phi$.

$$\begin{array}{rcl} \rho, t \models a & \text{iff} & \rho(t) = a \\ \rho, t \models \neg\phi & \text{iff} & \rho, t \not\models \phi \\ \rho, t \models \phi_1 \wedge \phi_2 & \text{iff} & \rho, t \models \phi_1 \text{ and } \rho, t \models \phi_2 \\ \rho, t \models \phi_1 \mathbf{U} \phi_2 & \text{iff} & \exists t' > t \cdot \rho, t' \models \phi_2 \text{ and } \forall t < t'' < t' \cdot \rho, t'' \models \phi_1 \\ \rho, t \models \phi_1 \mathbf{S} \phi_2 & \text{iff} & \exists t' < t \cdot \rho, t' \models \phi_2 \text{ and } \forall t' < t'' < t \cdot \rho, t'' \models \phi_1 \end{array}$$

Emptiness checking is decidable for LTL [4]. In fact, the untimed language of an LTL formula can be shown to be ω-regular. It can be used to express various properties like

- a is true somewhere in the future: $\mathbf{F}a = true\mathbf{U}a$
- a is forever true: $\mathbf{G}a = \neg\mathbf{F}\neg a$
- For each a a b occurs before the next a: $\mathbf{G}(a \Rightarrow (\neg a)\mathbf{U}b)$

These are quite useful for modelling reactive systems. But we also note that LTL can only specify qualitative properties i.e. relative occurences of events. It cannot specify the exact time gap between two events. This is the reason we define timed temporal logics.

Chapter 3

Timed temporal logics

Like LTL, timed temporal logics also use logical and modal operators, but additionally supply time constraints for these modal operators, as we shall see.

3.1 Preliminaries

Let $\mathbb{R}_0$ be the set of non-negative reals. An interval is a convex subset of $\mathbb{R}_0$, bounded by non-negative integer constants or ∞. The left and right ends of the interval might be open or closed. Let $\langle a, b\rangle$ denote a generic interval whose ends may be open or closed. An interval is *bounded* is it does not extend to ∞. It is called *singular* (or *punctual*) if it is of the form $[a, a]$ for some non-negative integer a.
We denote by $\mathbb{Z}I$ the set of all intervals. The set of non-punctual intervals is denoted by $\mathbb{Z}IExt$, and the set of all bounded intervals by $Bd\mathbb{Z}I$.

3.2 Metric Temporal Logic (MTL)

MTL is obtained by adding timing constraints to the Until and Since modal operators of LTL. The timing constraints are expressed as a set of allowed intervals, denoted by Iv. The resulting logic is $IvMTL[\mathbf{U}_I, \mathbf{S}_I]$.

3.2.1 Syntax

Let $I \in Iv$ and $a \in \Sigma$. Then the syntax for $IvMTL[\mathbf{U}_I, \mathbf{S}_I]$ is defined as follows

$$\phi ::= a \mid \phi \wedge \phi \mid \neg\phi \mid \phi \mathbf{U}_I \phi \mid \phi \mathbf{S}_I \phi$$

One can also define unary modal operators - future: $\mathbf{F}_I\psi = true\mathbf{U}_I\psi$ and past: $\mathbf{P}_I\psi = true\mathbf{S}_I\psi$
Well known classes of MTL are obtained by defining the class of intervals Iv:

- Full MTL, for $Iv = \mathbb{Z}I$
- Unary MTL, for $Iv = \mathbb{Z}I$ and only unary modal operators, i.e. $\mathbb{Z}IMTL[\mathbf{F}_I, \mathbf{P}_I]$
- Metric Interval Temporal Logic (MITL), for $Iv = \mathbb{Z}IExt$
- Bounded MTL, for $Iv = Bd\mathbb{Z}I$
- Suppose $\mathbb{Z}I^k$ denotes intervals of the form $\langle i, j\rangle$ or $\langle i, \infty)$ where i, j are non-negative integers such that $i, j \leq k$. The resulting logic is denoted by $MTL[\mathbf{U}_I, \mathbf{S}_I]^k$
- We can put multiple restrictions to get different subclasses of the full MTL in this way.

Define $MaxInt(\phi)$ to be the largest integer contant (apart from ∞) appearing in the interval constraints of formula ϕ.

3.2.2 Semantics

Given a pointwise timed word $\rho = (\overline{\sigma}, \overline{\tau})$, we define semantics for when ρ models formula ϕ at point $i \in dom(\rho)$

$\rho, i \models a$	iff	$\sigma_i = a$
$\rho, i \models \neg\phi$	iff	$\rho, i \not\models \phi$
$\rho, i \models \phi_1 \wedge \phi_2$	iff	$\rho, i \models \phi_1$ and $\rho, i \models \phi_2$
$\rho, i \models \phi_1 \mathbf{U}_I \phi_2$	iff	$\exists j > i \cdot \rho, j \models \phi_2$ and $\tau_j - \tau_i \in I$ and $\forall i < k < j \cdot \rho, k \models \phi_1$
$\rho, i \models \phi_1 \mathbf{S}_I \phi_2$	iff	$\exists j < i \cdot \rho, j \models \phi_2$ and $\tau_i - \tau_j \in I$ and $\forall j < k < i \cdot \rho, k \models \phi_1$

As earlier, we say ρ models ϕ iff $\rho, 1 \models \phi$. The semantics for continuous timed MTL are analogous, and presented below for completeness. In this case, we say ρ models ϕ iff $\rho, 0 \models \phi$.

$\rho, t \models a$	iff	$\rho(t) = a$
$\rho, t \models \neg\phi$	iff	$\rho, t \not\models \phi$
$\rho, t \models \phi_1 \wedge \phi_2$	iff	$\rho, t \models \phi_1$ and $\rho, t \models \phi_2$
$\rho, t \models \phi_1 \mathbf{U}_I \phi_2$	iff	$\exists t' > t \cdot \rho, t' \models \phi_2$ and $t' - t \in I$ and $\forall t < t'' < t' \cdot \rho, t'' \models \phi_1$
$\rho, t \models \phi_1 \mathbf{S}_I \phi_2$	iff	$\exists t' < t \cdot \rho, t' \models \phi_2$ and $t - t' \in I$ and $\forall t' < t'' < t \cdot \rho, t'' \models \phi_1$

3.3 Freeze logics

These logics specify timing constraints by putting conditions on special "freeze variables". Freeze variables remember the timestamp when a particular subformula is evaluated, and then can be compared to a future time instance, or other freeze variables. One such logic is Timed Linear Time Temporal Logic (TPTL) [1]. We shall not work with TPTL as far as this report is concerned, so we skip defining the syntax and semantics here. Details can be looked up in [1].

In their full generality, $MTL[\mathbf{U}_I, \mathbf{S}_I]$ and $TPTL[\mathbf{U}, \mathbf{S}]$ are undecidable even for finite timed words. Restrictions can be imposed to get decidable sub-logics [5]. Restrictions like using bounded intervals, or non-punctual intervals can result in decidability. Considering unary modalities also decreases the power of full MTL and TPTL. Hence it is very interesting to study these various fragments.
This report is mainly concerned with the expressive power of such fragments, which we start with in the next chapter.

Chapter 4

Expressiveness

4.1 Introduction

Each formula of a timed logic $\mathcal{P}$ defines a timed language. We define the language $\mathcal{L}(\phi)$ of formula $\phi \in \mathcal{P}$ as $\{\rho \ : \ \rho \models \phi\}$, both for pointwise and continuous interpretations. Let $\mathcal{L}(\mathcal{P})$ denote the set of languages definable in logic $\mathcal{P}$. We then say that a logic $\mathcal{P}_1$ is at least as expressive as logic $\mathcal{P}_2$ if $\mathcal{L}(\mathcal{P}_2) \subseteq \mathcal{L}(\mathcal{P}_1)$. By considering the sets of definable languages of a logic, we can define strictly more expressive, equally expressive and other such relations between logics.
Full versions of MTL and TPTL are undecidable, so it makes sense to study decidable fragments. But to create a fragment, we restrict the power of the logic. So it becomes interesting to know how these various relaxations affect expressive powers of logics. Alur and Henzinger proved [1, 2] that TPTL is at least as expressive as MTL. The reverse question had been open for many years, and was finally proved for pointwise time in [6] using Ehrenfeucht-Fraïssé games for MTL, which we shall describe shortly.
Various questions regarding relative expressiveness of fragments of MTL and TPTL remain open, both in the pointwise and continuous cases. Those that were proved had extremely complex proofs. But using the EF theorem for timed temporal logics, first proved in [6], one can hope to find simple game theoretic proofs of many of these results. We have a few such proofs for hitherto unknown results, that we shall present in this chapter. From hereon, we use $IvMTL^c[\mathbf{U}_I, \mathbf{S}_I]$ to denote MTL interpreted over continuous timed words.

4.2 Ehrenfeucht-Fraïssé games

Game theory has become an important part of computer science in general. Ehrenfeucht-Fraïssé games (EF games) have been useful for studying expressiveness of logics for quite some time. Classically, EF games have been used to prove such properties for first order logic [7]. Etessami and Wilke defined EF games for LTL, and used them to show an Until hierarchy for LTL [3]. These were further extended to MTL in [6]. We shall now define these EF games.

4.3 EF games for $IvMTL[\mathbf{U}_I, \mathbf{S}_I]$

We first deal with pointwise MTL. Let Iv be a set of allowed intervals. A k-round $IvMTL[\mathbf{U}_I, \mathbf{S}_I]$-EF game is played between two players called *Spoiler* and *Duplicator* on a pair of timed words ρ_0 and ρ_1. A configuration of the game at any point is a pair of positions (i_0, i_1) with $i_0 \in dom(\rho_0)$ and $i_1 \in dom(\rho_1)$. A configuration is called partially isomorphic, denoted $isop(i_0, i_1)$ iff $\sigma_{i_0} = \sigma_{i_1}$.
From a starting configuration (i_0, i_1), the game is defined inductively on k. One of the *Spoiler* and *Duplicator* will eventually win the game. The base case, a 0-round EF game is won by the *Duplicator* iff $isop(i_0, i_1)$. The $k+1$ round round game is played by first playing one round from the starting position. Either the *Spoiler* wins the round, and the game is terminated (means the *Spoiler* wins the game) or the *Duplicator* wins the round, and a new configuration occurs. Then a k round game is similarly played from the new configuration. The *Duplicator* wins the game only if it wins all rounds. Let us now describe one round of play from the starting configuration (i_0, i_1).

- At the start, if $\neg isop(i_0, i_1)$, the *Spoiler* wins the round. Otherwise,
- The *Spoiler* chooses one of the words by choosing $\delta \in \{0, 1\}$. Then $\overline{\delta}$ is the other word. The spoiler also chooses either a $\mathbf{U}_I$ or $\mathbf{S}_I$ move, along with the interval $I \in Iv$, which the *Duplicator* knows. The rest of the round is played in two parts.

$\mathbf{U}_I$ move.

- *Part I:* The *Spoiler* chooses a position i'_δ such that $i_\delta < i'_\delta \leq \#\rho_\delta$ and $(\tau_\delta[i'_\delta] - \tau_\delta[i_\delta]) \in I$.
- The *Duplicator* responds (note that *Duplicator* knows I already) by choosing $i'_{\overline{\delta}}$ in the other word such that $i_{\overline{\delta}} < i'_{\overline{\delta}} \leq \#\rho_{\overline{\delta}}$ and $(\tau_{\overline{\delta}}[i'_{\overline{\delta}}] - \tau_{\overline{\delta}}[i_{\overline{\delta}}]) \in I$. If the *Duplicator* cannot find such a position, the *Spoiler* wins the round (and the game). Else the game continues to *Part II*.
- *Part II: Spoiler* chooses to play either **F**-part or **U**-part.
  - **F**-part: the round ends with the configuration (i'_0, i'_1).
  - **U**-part: The *Spoiler* checks whether $i'_{\overline{\delta}} - i_{\overline{\delta}} = 1$. If *yes*, the round ends with the configuration (i'_0, i'_1). Otherwise, *Spoiler* chooses a position $i''_{\overline{\delta}}$ in $\rho_{\overline{\delta}}$ such that $i_{\overline{\delta}} < i''_{\overline{\delta}} < i'_{\overline{\delta}}$. The *Duplicator* responds by choosing i''_δ in ρ_δ such that $i_\delta < i''_\delta < i'_\delta$. The round ends with the configuration (i''_0, i''_1). If the *Duplicator* is unable to choose an i''_δ, the game ends and the *Spoiler* wins.

$\mathbf{S}_I$ move. This is symmetric to the $\mathbf{U}_I$ move. The difference is that in *Part I* the *Spoiler* chooses a point in the past instead of future, as does the *Duplicator* . In *Part II*, the *Spoiler* can choose to play the **P**-part or the **S**-part.

We also make the following observation. Suppose the *Spoiler* chooses a $\mathbf{U}_I$ move, and moves to the successor position on the word δ. Then the *Duplicator* is forced to move to a successor position on $\overline{\delta}$, because if not, the *Spoiler* will play the **U**-part and the *Duplicator* will not be able to move. On the other hand, if the *Spoiler* chooses a non-successor position, the *Duplicator* can still choose a successor position if it wishes to. Secondly, we can also define $IvMTL[\mathbf{F}_I, \mathbf{P}_I]$ games, or $IvMTL[\mathbf{U}_I]$ games etc. Correspondingly, the moves that the *Spoiler* can play get restricted.

Definition 1. *Given two timed words ρ_0, ρ_1 and $i_0 \in dom(\rho_0)$, $i_1 \in dom(\rho_1)$, we define*

- **Game equivalence:** $(\rho_0, i_0) \approx_k^{Iv} (\rho_1, i_1)$ *iff for every k-round $IvMTL[\mathbf{U}_I, \mathbf{S}_I]$ EF-game over the words ρ_0, ρ_1 starting from the configuration (i_0, i_1), the Duplicator always has a winning strategy.*
- **Formula equivalence:** $(\rho_0, i_0) \equiv_k^{Iv} (\rho_1, i_1)$ *iff for every $IvMTL[\mathbf{U}_I, \mathbf{S}_I]$ formula ϕ of modal depth $\leq k$, $\rho_0, i_0 \models \phi \Leftrightarrow \rho_1, i_1 \models \phi$.*

We can finally state the $IvMTL[\mathbf{U}_I, \mathbf{S}_I]$ EF theorem. The proof is a straight forward extension of the proof of LTL EF Theorem of [3], and is given in [6]. What is to be noted is that to restrict the number of modal depth $\leq n$ formulas to finitely many (upto equivalence, given the words ρ_0 and ρ_1), we consider only those with the $MaxInt$ bounded by k. Here k is the smallest integer larger than $max(time(\rho_0), time(\rho_1))$. Lemma 1 then helps us prove the EF theorem for all formulas with modal depth $\leq n$.

Lemma 1. *For any $\phi \in MTL[\mathbf{U}_I, \mathbf{S}_I]$ and any integer k, let $[\phi]^k$ denote the formula obtained by replacing in ϕ any occurence of any constant $c > k$ (or ∞) by k. Let ρ be a timed word and let integer $k > \tau_{\#\rho}$. Then $\forall i \cdot \rho, i \models \phi \Leftrightarrow \rho, i \models [\phi]^k$. Hence given a word, it is not possible for a formula to distinguish contants (or ∞) greater than $\tau_{\#\rho}$.*

Proof. The statement is quite intuitive, and the proof can be done by a simple induction on the modal depth of a given formula ϕ. One needs to consider appropriate cases, though. □

Theorem 1. $(\rho_0, i_0) \approx_n^{Iv} (\rho_1, i_1)$ *iff* $(\rho_0, i_0) \equiv_n^{Iv} (\rho_1, i_1)$

The proof can be found in [6].

4.3.1 Proving expressiveness results using EF games

Given logics $\mathcal{P}_1$ and $\mathcal{P}_2$. We wish to show that $\mathcal{P}_1$ contains a formula whose language cannot be expressed by a $\mathcal{P}_2$ formula. To do this, we first guess a candidate formula $\phi_1 \in \mathcal{P}_1$. Then for each $n \geq 0$, we construct a pair or words $\mathcal{A}_n$ and $\mathcal{B}_n$ such that $\mathcal{A}_n \models \phi_1$ and $\mathcal{B}_n \not\models \phi_1$. Further, we show that for every n, $(\mathcal{A}_n, 1) \approx_n^{Iv} (\mathcal{B}_n, 1)$ by finding a *Duplicator* strategy in the $\mathcal{P}_2$ EF game. Then this implies that the formula ϕ_1 has no equivalent formula in $\mathcal{P}_2$. For if there did exist a formula ϕ_2 of modal depth k such that $\phi_1 \equiv \phi_2$, then $(\mathcal{A}_n, 1) \approx_k^{Iv} (\mathcal{B}_n, 1) \Rightarrow \mathcal{A}_n \equiv_k^{Iv} \mathcal{B}_n$, but this is a contradiction as ϕ_2 should be able to distinguish between $\mathcal{A}_n$ and $\mathcal{B}_n$.
We note that this kind of reasoning actually appeals to only half the EF theorem, i.e. the one sided implication $(\rho_0, i_0) \approx_n^{Iv} (\rho_1, i_1) \Rightarrow (\rho_0, i_0) \equiv_n^{Iv} (\rho_1, i_1)$. Using this approach, various previously unknown results (along with vastly simpler proofs of previously known results) are proved in [6]. Here, we present a few new examples.

4.3.2 Results using pointwise EF games

Theorem 2. $MITL[\mathbf{F}_I, \mathbf{P}_I] \not\subseteq MITL[\mathbf{F}_I]$

Proof. We consider the $MITL[\mathbf{F}_I, \mathbf{P}_I]$ formula $\phi = \mathbf{F}_{[0,\infty)}[b \wedge \mathbf{F}_{(0,1)}(c \wedge \neg \mathbf{P}_{(0,1)} c)]$.
We shall prove the result by giving a *Duplicator* winning strategy for an n-round $MITL[\mathbf{F}_I]$ EF game on the words $\mathcal{A}_n$ and $\mathcal{B}_n$, which are described as follows. Let us also suppose that the *Spoiler* is allowed to use integer constants upto a fixed k only. The *Spoiler* can choose ∞ as one of the bounds, of course.

$\mathcal{A}_n$ is defined as follows:

0	1	$\cdots$	$m-1$	m	m	$m+0.7$	$m+0.7+\epsilon$	$m+0.7+2\epsilon$	$m+0.7+3\epsilon$	$\cdots$	$m+0.7+m\epsilon$
d	d	$\cdots$	d	b	c	c	c	c	c	$\cdots$	c

$\mathcal{B}_n$ is defined as follows:

0	1	$\cdots$	$m-1$	m	$m+0.1$	$m+0.7$	$m+0.7+\epsilon$	$m+0.7+2\epsilon$	$m+0.7+3\epsilon$	$\cdots$	$m+0.7+m\epsilon$
d	d	$\cdots$	d	b	c	c	c	c	c	$\cdots$	c

The upper line indicates time-stamps, and the lower line indicates events that occur at those time stamps. $m = (n+1)(k+1)$. Note that the word $\mathcal{A}_n$ is weakly monotonic. For the stricly monotonic case, a different example is required. ϵ is a small positive real number. We can take $\epsilon = \frac{1}{(n+10)^m}$. We note that $\mathcal{A}_n \not\models \phi$ and $\mathcal{B}_n \models \phi$.
The *Duplicator* winning strategy can be described as below:

1. The *Spoiler* can start out on either word. But as long as he continues to move to one of the d's, the *Duplicator* can copy those moves exactly.
2. The only way for the *Spoiler* to introduce a difference is by quickly coming to a time point $\geq m$.
3. Since the *Spoiler* cannot choose constants greater than k, he has to choose an interval $\langle j, \infty)$ if he wants to jump to m.
4. Once he does that, the only interval that can be chosen is $\langle 0, 1)$ as both words end before $m+1$. In either case, he can only introduce a difference of one position. For instance, if he chooses to play on the word $\mathcal{B}_n$ the interval $(m, m+1)$, and moves to the time point $m+0.1$, the *Duplicator* will move to the point $m+0.7$ on $\mathcal{A}_n$. That leaves $n-1$ rounds to play, and there are enough c to the right now to exhaust all these rounds. If the *Spoiler* moves a distance of 2 or more on either word, the *Duplicator* catches up, so the *Spoiler* is forced to move 1 step at a time on $\mathcal{B}_n$. The rounds will be exhausted before the end is reached.

□

We now establish an MTL hierarchy based on $MaxInt$ of an MTL formula. The necessary definitions are in Chapter 3 Section 3.2.1. It is clear that $MTL[\mathbf{U}_I, \mathbf{S}_I]^k \subseteq MTL[\mathbf{U}_I, \mathbf{S}_I]^{k+1}$. We show that this containment is actually strict. Thus, allowing larger constants in constraints actually increases the power of the logic. What is also very interesting is that the result changes in case of continuous timed MTL. The result for continuous MTL is shown in Claim 5.

Theorem 3. $MTL[\mathbf{U}_I, \mathbf{S}_I]^{k+1} \not\subseteq MTL[\mathbf{U}_I, \mathbf{S}_I]^k$

Proof. We consider the $MTL[\mathbf{U}_I, \mathbf{S}_I]^{k+1}$ formula $\phi = a\mathbf{U}_{[k+1,\infty)}(a\mathbf{S}_{[k,k+1)}a)$.
We prove the result by giving a *Duplicator* winning strategy for an n-round $MTL[\mathbf{U}_I, \mathbf{S}_I]^k$ EF game on the words $\mathcal{A}_n$ and $\mathcal{B}_n$, which are described as follows.

$\mathcal{A}_n$ is defined as follows:

0	0.6	$k-0.3$	$k+1$	$2k+1$	$3k+1$	$\cdots$	$nk+1$
a	a	a	a	b	b	$\cdots$	b

$\mathcal{B}_n$ is defined as follows:

0	0.4	$k-0.3$	$k+1.6$	$2k+1.6$	$3k+1.6$	$\cdots$	$nk+1.6$
a	a	a	a	b	b	$\cdots$	b

The upper line indicates time-stamps, and the lower line indicates events that occur at those time stamps. Clearly, $\mathcal{A}_n \models \phi$ and $\mathcal{B}_n \not\models \phi$.
The fact that the *Duplicator* has a winning strategy can be very easily shown for these two words. First, we note that $Untime(\mathcal{A}_n) = Untime(\mathcal{B}_n) = aaaab^{n-1}$. This means that $dom(\mathcal{A}_n) = dom(\mathcal{B}_n) = dom$ (say). Then

$$\forall i \forall j \in dom \cdot \{|\tau_{\mathcal{A}_n}[i] - \tau_{\mathcal{A}_n}[j]| \in (p,q) \wedge |\tau_{\mathcal{B}_n}[i] - \tau_{\mathcal{B}_n}[j]| \in (p,q)\} \vee \{|\tau_{\mathcal{A}_n}[i] - \tau_{\mathcal{A}_n}[j]| > k \wedge |\tau_{\mathcal{B}_n}[i] - \tau_{\mathcal{B}_n}[j]| > k\}$$

In short, the time gaps between corresponding points in $\mathcal{A}_n$ and $\mathcal{B}_n$ belong to the same open interval, or are both greater than k. The simple *Duplicator* strategy is to follow the *Spoiler* moves position to position. Since the *Spoiler* can only use contants $\leq k$, the *Duplicator* can actually always do that. Thus the *Duplicator* wins. ☐

4.4 EF games for $IvMTL^c[\mathbf{U}_I, \mathbf{S}_I]$

These are similar to pointwise $MTL[\mathbf{U}_I, \mathbf{S}_I]$ EF games. But they are played with the continuous semantics over two timed words. The *Spoiler* and *Duplicator* can now place their pebbles on any point on the real line (instead of just the event points).
From a starting configuration (t_0, t_1), the game is defined inductively on the number of rounds. The *Duplicator* wins the 0-round game iff $isop(t_0, t_1)$. The $n+1$ round game is played by first playing one round from the starting position. Either the *Spoiler* wins the round and the game ends, or a configuration (t'_0, t'_1) results and an n round game is played from the new configuration. The *Duplicator* must win each round to win the game. Suppose the two words are ρ_0, ρ_1 and the starting configuration is (t_0, t_1). Then the play in a round proceeds as below.

- At the start, if $\neg isop(t_0, t_1)$, the *Spoiler* wins the round. Otherwise,
- The *Spoiler* chooses one of the words by choosing $\delta \in \{0, 1\}$. Then $\overline{\delta}$ is the other word. The spoiler also chooses either a $\mathbf{U}_I$ or $\mathbf{S}_I$ move, along with the interval $I \in Iv$, which the *Duplicator* knows. The rest of the round is played in two parts.

$\mathbf{U}_I$ move.

- *Part I:* The *Spoiler* chooses a position t'_δ such that $t_\delta < t'_\delta \leq time(\rho_\delta)$ and $(t'_\delta - t_\delta) \in I$.
- The *Duplicator* responds (note that *Duplicator* knows I already) by choosing $t'_{\overline{\delta}}$ in the other word such that $t_{\overline{\delta}} < t'_{\overline{\delta}} \leq time(\rho_{\overline{\delta}})$ and $(t'_{\overline{\delta}} - t_{\overline{\delta}}) \in I$. If the *Duplicator* cannot find such a time instance, the *Spoiler* wins the round (and the game). Else the game continues to *Part II*.

– *Part II: Spoiler* chooses to play either **F**-part or **U**-part.

  ◦ **F**-part: the round ends with the configuration (t'_0, t'_1).
  ◦ **U**-part: The *Spoiler* chooses a position $t''_{\bar{\delta}}$ in $\rho_{\bar{\delta}}$ such that $t_{\bar{\delta}} < t''_{\bar{\delta}} < t'_{\bar{\delta}}$. The *Duplicator* responds by choosing t''_δ in ρ_δ such that $t_\delta < t''_\delta < t'_\delta$. The round ends with the configuration (t''_0, t''_1). If the *Duplicator* is unable to choose a t''_δ, the game ends and the *Spoiler* wins.

S$_I$ **move**. This is symmetric to the **U**$_I$ move. The difference is that in *Part I* the *Spoiler* chooses a point in the past instead of future, as does the *Duplicator* . In *Part II*, the *Spoiler* can choose to play the **P**-part or the **S**-part.

Just as in the pointwise case, we can restrict the logic in various ways and restrict moves of the *Spoiler* .

Definition 2. *Given two timed words ρ_0, ρ_1 and $t_0 \le time(\rho_0)$, $t_1 \le time(\rho_1)$, we define*

◦ **Game equivalence:** $(\rho_0, t_0) \approx_k^{c,Iv} (\rho_1, t_1)$ *iff for every k-round $IvMTL^c[\mathbf{U}_I, \mathbf{S}_I]$ EF-game over the words ρ_0, ρ_1 starting from the configuration (t_0, t_1), the Duplicator always has a winning strategy.*

◦ **Formula equivalence:** $(\rho_0, t_0) \equiv_k^{c,Iv} (\rho_1, t_1)$ *iff for every $IvMTL^c[\mathbf{U}_I, \mathbf{S}_I]$ formula ϕ of modal depth $\le k$, $\rho_0, t_0 \models \phi \Leftrightarrow \rho_1, t_1 \models \phi$.*

We state the $IvMTL^c[\mathbf{U}_I, \mathbf{S}_I]$ EF theorem. The proof is similar to that for the pointwise case.

Theorem 4. $(\rho_0, t_0) \approx_n^{c,Iv} (\rho_1, t_1)$ *iff* $(\rho_0, t_0) \equiv_n^{c,Iv} (\rho_1, t_1)$

Proving expressiveness results for continuous timed logics involves ideas similar to the pointwise logic. But now the proofs become more difficult, as we're dealing with continuous timed words. Here we present two new results. One of them demonstrates the use of EF games. The second result is actually about the equivalence of two logics. With this, we also note that EF games can be used to give *negative* results. They aren't useful for showing actual equivalence of two logics.

4.4.1 Some expressiveness results for continuous timed logics

The following result is an interesting one, because it shows that the MTL hierarchy which we showed in Claim 3 actually collapses when we move to continuous logics. This highlights a key difference between pointwise and continuous logics.

Theorem 5. *For a positive integer $k \ge 1$, $MTL^c[U_I, S_I]^{k+1} = MTL^c[U_I, S_I]^k$.*

Proof. It is clear that $MTL^c[U_I, S_I]^k \subseteq MTL^c[U_I, S_I]^{k+1}$. We shall now show that any formula $\phi \in MTL^c[U_I, S_I]^{k+1}$ can be expressed in $MTL^c[U_I, S_I]^k$.

We prove this for Until formulae. The proof for Since holds similarly. The proof is by induction on the modal depth of the formula. The proof for the base case is similar to that for the induction step.

Let $\phi = \varphi U_I \psi$, with both φ and ψ having a modal depth $\le n$. The only interesting case is when the interval I falls in one of the following cases:

Case I: $I = \langle k+1, \infty)$
Then φ must hold at point k. And then, ψ must be true after that point at a distance of 1 or more. So the formula $\phi' = \varphi U_{[k,k]}(\varphi \wedge \varphi U_{\langle 1,\infty)} \psi)$ is true iff ϕ is true. Also, since both φ and ψ had modal depth $\le n$, they are expressible in $MTL^c[U_I, S_I]^k$. This completes the induction for this case.

Case II: $I = [k+1, k+1]$
$\phi'' = \varphi U_{[k,k]}(\varphi \wedge \varphi U_{[1,1]} \psi)$ is equivalent to ϕ.

Case III: $I = \langle j, k+1\rangle$ for some integer $j < k$
Then we take $\phi''' = \varphi U_{\langle j,k]} \psi \ \vee \ \varphi U_{[k,k]}(\varphi \wedge \varphi U_{(0,1\rangle} \psi)$.

In all cases, we have been able to get rid of the $k+1$ in interval constraints. The induction step for an $n+1$ modal depth formula is complete. □

Note that the key point here was that since we're looking at continuous MTL, we know there is always a point k. We can then put conditions on it and go as further ahead from k as we want, using constants no larger than k.

In the next result, we note that the past by itself does provide more power than the fully powerful future fragment of MTL. In short, if we allow all kinds of intervals for the $\mathbf{U}_I$ modality, we still can't capture some properties of the past $\mathbf{P}$ modality.

Theorem 6. $MITL^c[\mathbf{F}_B, \mathbf{P}] - MTL^c[\mathbf{U}_I] \neq \emptyset$.

Proof. Consider the $MTL^c[\mathbf{F}_B, \mathbf{P}]$ formula

$$\phi = \mathbf{F}_{(0,1)}(a \wedge \mathbf{P}a)$$

It says there are atleast 2 a's in (0,1). Consider the following words used in an n-round game, where spoiler's max constant is k.

1. Word A: In (0,1), there are two a's, one at $0.5 - \epsilon$ and the second at 0.5. In (1,2), there are n a's between 1 and 1.1. Assume these are placed at $1.01, 1.01 + \epsilon, \ldots, 1.01 + n\epsilon$. In $(2, nk + 1)$, there are no events.

2. Word B : same as word A in $(1, nk + 1)$. In interval (0,1), there is a single a at 0.5.

Clearly, $A \models \phi$ and $B \not\models \phi$. We now play a $\mathbf{U}_I$ game.

1. Suppose *Spoiler* is on A and *Duplicator* is on B. If *Spoiler* chooses the a at $0.5 - \epsilon$ with an $\mathbf{F}_{(0,1)}$ move, then *Duplicator* has to choose the a at 0.5 in B. If *Spoiler* chooses another $\mathbf{F}_{(0,1)}$ and comes to the a at 0.5, then *Duplicator* will come to the a at 1.01. Note that in between the points $0.5 - \epsilon, 0.5$ of A as well as $0.5, 1.01$ of B, there are no a's. Further, from any point in between, there is no a at a precise distance. If *Spoiler* had moved to the a at 1.01 in A instead of choosing the a at 0.5, then *Duplicator* will choose the a at $0.01 + \epsilon$. Then, in both cases, there is an a in between; moreover, from any of the points in between, there are no a's at a precise distance, but ofcourse, there are a's at imprecise intervals.

2. The case when *Spoiler* is on A and starts with the second a is easier, in this case, *Duplicator* chooses his only a in (0,1). The words are identical from now on. The only interesting point here is that now, *Spoiler* is on the second a, and hence has an a in between 0 and his current point; *Duplicator* has none. But *Spoiler* , if at all, he wants to play an until move, can only pick a point in B, and that will be a $\neg b$ in (0.0.5). To this, *Duplicator* will answer picking a point in $(0.5 - \epsilon, 0.5)$.

3. If *Spoiler* starts out on B at the first a, *Duplicator* will start out on A at the first a at $0.5 - \epsilon$. (*Duplicator* starting on the second a at 0.5 would be a problem for him, as *Spoiler* can pick the a in between at $0.5 - \epsilon$). If *Spoiler* makes an $F_{(0,1)}$ and comes to 1.01, the *Duplicator* will come to the a at 0.5. Note that there are no intermediate a's, and from none of the points in between, a precise move will give an a. The game can then continue as in point 1 above.

Clearly, *Duplicator* wins. □

4.5 Using EF games to show logic equivalence

So far, we have seen how to use EF games to establish the fact that logic $\mathcal{A}$ contains a formula which logic $\mathcal{B}$ does not. This is used to show strict containment or that a logic is not contained in another. In this section, we present a new approach using the EF theorem which actually establishes equivalence of two logics. We shall appeal to this approach in future chapters. They key is the following lemma. $\mathcal{K}$ is a set of models and $\mathcal{A}$ is a logic with finitely many atomic propositions.

Lemma 2. *There exists a formula $\phi \in \mathcal{A}$ which accepts exactly the models in $\mathcal{K}$ iff* $\exists\, n_0 \cdot \forall A \forall B \cdot (A \in \mathcal{K} \wedge B \notin \mathcal{K} \Rightarrow A \approx_{n_0}^{\mathcal{A}} B)$*, where the EF game is played for logic $\mathcal{A}$.*

Proof. (⇒) If there exists $\phi \in \mathcal{A}$, and it has depth k, then in a k-round EF game (for logic $\mathcal{A}$) on a pair of words, one from $\mathcal{K}$ and the other not from $\mathcal{K}$, *Spoiler* will always win. We simply choose $n_0 = k$.
(⇐) We consider the following formula
$\psi = \bigvee_{A \in \mathcal{K}} (\bigwedge_{A \models \alpha,\ depth(\alpha) \leq n_0} \alpha)$. Since $\mathcal{K}$ could be an infinite class, ψ could be an infinite formula. But note that since logic $\mathcal{A}$ has only finitely many atomic propositions, the set $T = \{\alpha \in \mathcal{A} \mid depth(\alpha) \leq n_0\}$ is finite, and hence all disjuncts of ψ are finite. Additionally, each disjunct is a subset of T, so the number of possible unique disjuncts can be no more than $|2^T|$. So we modify ψ and retain only unique disjuncts to obtain a finite formula $\phi \in \mathcal{A}$. Using the EF theorem for $\mathcal{A}$, it is easy to see that ϕ accepts exactly the set of models $\mathcal{K}$.

□

The approach to prove equivalence can be described as follows. Suppose we wish to show that logic $\mathcal{B}$ is equivalent to logic $\mathcal{A}$, where both logics have finitely many atomic propositions. Consider an arbitrary $\psi \in \mathcal{A}$, with depth n. Also consider arbitrary models A and B such that $A \models \psi$ and $B \not\models \psi$. Then in an n-round $\mathcal{A}$ EF game on A and B, *Spoiler* has a winning strategy. We use this winning strategy in the $\mathcal{A}$ EF game to come up with a *Spoiler* winning strategy in a $\mathcal{B}$ EF game, in $f(n)$ rounds, where the function f depends only on ψ, and is independent of the models A and B. We then appeal to Lemma 2 with $n_0 = f(n)$ and $\mathcal{K} = \{A \mid A \models \psi\}$ to show existence of an equivalent formula $\phi \in \mathcal{B}$.
Note that the crucial part is showing the existence of such a function f, after which, equivalence follows.

Chapter 5

Timed and untimed past

In this chapter, we try to tackle the major open problem that compares timed and untimed past. The open question is whether $MTL^c[\mathbf{U}_I, \mathbf{S}_I]$ is strictly stronger than $MTL^c[\mathbf{U}_I, \mathbf{S}]$. The difference is that in the former, timed moves to the past are allowed, where as in the latter, if a move to the past has to be made, then no timing constraints can be utilized. The full open problem doesn't easily admit an EF game separation result, and we shall discuss more about it in Section 5.2.

But before tackling the main result, there are a host of other interesting questions of the same flavour, that can be asked for sub-logics (lesser modalities or lesser intervals or both). We now start looking at a few such results that give some insight into how the strength of untimed past varies with modalities and the allowed intervals.

As shall be observed in this chapter, if we have separation results, they will be quite similar in technique to the ones introduced in Chapter 4. But for results which prove equivalence of logics, we shall use either special constraints on the logic (like boundedness) or a game translation.

5.1 Results with sub-logics

The simplest case to consider was the unary fragment of MTL. We would like to know if $MTL^c[\mathbf{F}_I, \mathbf{P}_I]$ is strictly stronger than its counterpart with untimed past. Before that, we consider the same problem with bounded future fragment, and negation free (only assertive) formulas.

5.1.1 Bounded and negation-free unary fragment

Theorem 7. $NF\ MTL^c[\mathbf{F}_B, \mathbf{P}_I] = NF\ MTL^c[\mathbf{F}_B, \mathbf{P}]$*, where NF means that negations are not allowed in the logic.*

We illustrate this theorem using an example. Consider $\phi = F_{[2,3]}(a \ \wedge \ P_{[0,1]}(b \ \wedge \ F_{[1,2]}c))$. Consider a model A such that $A \models \phi$. Clearly, neither ϕ or any of its subformulas will ever be interpreted beyond the time point 5, so we can assume without loss of generality that the model is blank beyond 5. Note that such a bound can always be obtained for any formula with bounded future.

Now note that ϕ talks about three points in all - points where the following atomic events hold - a, b, c. Now depending on the relative positions of these three points, we can come up with and $MTL^c[\mathbf{F}_B, \mathbf{P}]$ formula that tries to simulate this behaviour. The basic idea is to map all behaviours to the interval $[0, 1]$. Suppose that in A, the three points being talked about are at the following locations (in order): 2.2, 1.8, 3.1. We need to assert that these points have a, b, c respectively. We order these points in the reverse order of fractional parts: 1.8, 2.2, 3.1. Now to capture this, we write down the following $MTL^c[\mathbf{F}_B, \mathbf{P}]$ formula $\phi_1 = F_{[1,1]}(P(F_{[1,1]}b \ \wedge \ P(F_{[2,2]}a \ \wedge \ P(F_{[3,3]}c))))$. Clearly this captures the required behaviour. Note that this also captures all behaviours in which the three points that ϕ talks about have the same ordering of fractional parts, and lie in the same unit intervals. So essentially 2.2, 1.8, 3.1 is equivalent to 2.4, 1.6, 3.3. Note that even these new set of points satisfy the constraints of ϕ, and are accepted by ϕ_1 as well.

There are two key ideas to prove the theorem. Firstly, that if the 3 (in general k) points that the formula ϕ talks about are placed in a certain configuration w.r.t. unit intervals and ordering of their fractional parts, then any other such placing which satisfies the same unit intervals and ordering of fractional parts is also a satisfiable instance of ϕ (this requires the fact that ϕ is negation-free), and a single ϕ_1 captures both of these (or all of these models with the same configuration). This is easy to see because the intervals allowed in the logic have integer end-points. The second idea to note is that the number of such configurations is finite, which again holds simply because the logic is totally bounded.

One last thing that remains to be shown is that if a formula of the form of ϕ_1 is such that $A \models \phi_1$, and there exists at least one other model A' with the same configuration as A such that $A' \models \phi$, then this implies that $A \models \phi$.

We must make the additional note here that if there is no explicit assertion at the level of some sub-formula, then we must simply add a $true$ predicate for completeness. So a formula $\phi = F_{[2,3]}(a \wedge P_{[0,1]}(F_{[1,2]}c))$ must be seen as $\phi = F_{[2,3]}(a \wedge P_{[0,1]}(true \wedge F_{[1,2]}c))$.

5.1.2 Bounded unary case with bounded models

This result is an interesting one. It compares bounded unary logics with timed and untimed past. Addtitionally, we shall prove this result only for bounded models - that is, models which end at a certain finite point for a given bound on the logic. We show that in this case, even with negations allowed, the two logics are equivalent.
Let us consider the logic $MTL^c[\mathbf{F}_B, \mathbf{P}_B]$, where the bounds for the future and past respectively are k and r. Given a formula ϕ in this logic, we wish to write an equivalent $MTL^c[\mathbf{F}_B, \mathbf{P}]$ formula.
Consider $\phi \in MTL^c[\mathbf{F}_B, \mathbf{P}_B]$. Let ϕ have a modal depth of n and its future and past bounds be k and r respectively. Then for this ϕ, we have the following lemma.

Lemma 3. *Take timed words A and B such that $A \models \phi$ and $B \not\models \phi$. Then in the $MTL^c[\mathbf{F}_B, \mathbf{P}_B]$ EF game between A and B, the Spoiler has a winning strategy in n rounds such that it can win by staying within the time point $y_0 = k \cdot n$.*

Proof. The fact that the *Spoiler* wins in n rounds follows from the EF theorem (Theorem 4) as ϕ distinguishes between A and B and has modal depth n.
Also, since with each move, the *Spoiler* or *Duplicator* can advance by at most k units, they can advance no more than $y_0 = k \cdot n$ units in the n round game. This completes the proof of the lemma. □

In view of Lemma 3, we shall only consider models that last upto $y_0 = k \cdot n$. So given $\phi \in MTL^c[\mathbf{F}_B, \mathbf{P}_B]$, we describe how to obtain an equivalent $MTL^c[\mathbf{F}_B, \mathbf{P}]$ formula. The key will be to use the approach described in Section 4.5. We already have an n round $MTL^c[\mathbf{F}_B, \mathbf{P}_B]$ EF game in which the *Spoiler* wins. Let us call this game $\mathcal{G}_1$. We shall obtain an $MTL^c[\mathbf{F}_B, \mathbf{P}]$ EF game $\mathcal{G}_2$ which would have the *Spoiler* winning in at most y_0 rounds. We would then have the existence of the required $MTL^c[\mathbf{F}_B, \mathbf{P}]$ formula by Lemma 2.

We now describe how to obtain $\mathcal{G}_2$ from $\mathcal{G}_1$. The idea is to restrict the game in $\mathcal{G}_2$ to the unit interval $[0, 1]$ where timed and untimed past are equivalent.

Obtaining the untimed game

We first note that in $\mathcal{G}_1$, the *Duplicator* must always remain in the same unit interval as *Spoiler* , else the *Spoiler* will make a timed past move and beat the *Duplicator* . Additionally, we can assume without loss of generality that the intervals with the *Spoiler* uses are either precise or of the form $\langle i, i+1 \rangle$. With this assumption, we note that with every round in $\mathcal{G}_1$, the fractional parts of the positions in either word either simultaneously increase or decrease.

We use these facts to play $\mathcal{G}_2$ (y_0 rounds). The *Spoiler* simulates moves of $\mathcal{G}_1$, but only their fractional parts. The *Duplicator* is obliged to follow by moving to the fractional part of the point that he would have moved to in $\mathcal{G}_1$, because if he didn't, the *Spoiler* will make a precise future move that would result in a sub-optimal configuration for the *Duplicator* .

Additionally, the *Duplicator* cannot go beyond the time point 1, otherwise there are enough rounds for the *Spoiler* to reach the end of the word and force the *Duplicator* to reach the end of the word (remember that the words are bounded, and do not extend beyond y_0).

5.1.3 Separation in the full unary case

The following result shows that the full unary MTL with untimed past cannot express a timed past property.

Theorem 8. $MITL^c[\mathbf{F}_B, \mathbf{P}_B] - MTL^c[\mathbf{F}_I, \mathbf{P}] \neq \emptyset$

Proof. The candidate formula is $\phi = \Box_{(1,2)}(b \Rightarrow P_{(0,1)}(b \wedge P_{(0,1)}s))$

Consider the following two words as models to be used while playing an n-round game, where spoiler's maximum constant is k.

1. Word $\mathcal{A}_n$: At point 0, s is true. In interval $[1,2]$, we have b at the points $1.2, 1.2+\epsilon, \cdots, 1.2+n\epsilon$. In the interval $[2,3]$, we have b at the points $2.2-\delta, 2.2+(\epsilon-\delta), 2.2+(2\epsilon-\delta), \cdots, 2.2+(n\epsilon-\delta)$. In the interval $[3,4]$, we have b at points $3.2-2\delta, 3.2+(\epsilon-2\delta), 3.2+(2\epsilon-2\delta), \cdots, 3.2+(n\epsilon-2\delta)$, and so on, in the interval $[nk, nk+1]$, we have b at points $nk+0.2-(nk-1)\delta, nk+0.2+(\epsilon-(nk-1)\delta), \cdots, nk+0.2+(n\epsilon-(nk-1)\delta)$. In the interval $[0,1]$, we have n b's between 0.2 and $0.2+\epsilon$, we have n b's between $0.2+\epsilon$ and $0.2+2\epsilon$, and so on, and we have n b's between $0.2+(n-1)\epsilon$ and $0.2+n\epsilon$. Each of these n b's have been placed so that any precise move from them will land at a $\neg b$. Thus, for instance, if we look at the first block of n b's between 0.2 and $0.2+\epsilon$, they are not placed at the points $0.2+(\epsilon-\delta), 0.2+(\epsilon-2\delta), \cdots, 0.2+(\epsilon-(nk-1)\delta)$. Similar is the case of the other blocks of b's in [0,1].
2. Word $\mathcal{B}_n$ has all the b's as in $\mathcal{A}_n$. In addition, there is an extra block of n b's between the points $0.2+n\epsilon$ and $0.2+(n+1)\epsilon$. Call the positions of this last block of b's as $k_1, k_2, \cdots, k_n$.
3. $\mathcal{A}_n$ and $\mathcal{B}_n$ differ only in the number of b's between $0.2+n\epsilon$ and $0.2+(n+1)\epsilon$. In $\mathcal{A}_n$, there are none, in $\mathcal{B}_n$, there are n of these at positions $k_1, k_2, \cdots, k_n$.
4. The key point to note is that both words are same in $[1, nk+1]$. Furthermore, in all intervals, for every b, there is a b in the next interval at $(0,1)$ from it. There are no two b's at a precise distance.

It is easy to see that $\mathcal{B}_n \models \varphi$ while $\mathcal{A}_n \not\models \varphi$. There are n b's in $[1,2]$. In $\mathcal{A}_n$, the b at $1.2+n\epsilon$ does not have a b at $(0,1)$ from it in the interval $[0,1]$. In $\mathcal{B}_n$, there is no issue, as the last block of b's in $[0,1]$ satisfy the requirement.

The fact that the *Duplicator* wins is easy to argue from here, and we skip the description of the winning stragegy for now. ☐

We have one additional result which shows that a stronger (untimed) past component **S** assumes some power that even unary timed past ($\mathbf{P}_I$) cannot capture. Note that the future fragment must be restricted to $\mathbf{F}_I$ in both cases, because $\mathbf{U}_I$ is too powerful, and coupled with $\mathbf{P}_I$, it can do much more.

Theorem 9. $MTL^c[\mathbf{F}_I, \mathbf{S}] - MTL^c[\mathbf{F}_I, \mathbf{P}_I] \neq \emptyset$

Proof. We consider the $MTL^c[\mathbf{F}_I, \mathbf{S}]$ formula $\phi = \Box_{(0,1)}(a \Rightarrow \neg S(F_{[1,1]}a))$.
We prove the result by giving a *Duplicator* winning strategy for an n-round $MTL^c[\mathbf{F}_I, \mathbf{P}_I]$ EF game on the words $\mathcal{A}_n$ and $\mathcal{B}_n$, which are described as follows. Here $0 < \delta < \epsilon << 1$

$\mathcal{A}_n$ is defined as follows: Place a's at the points $0.2, 0.2+\epsilon, 0.2+2\epsilon, \cdots, 0.2+2n\epsilon$. Also place a's at the points $0.2-\delta, 0.2+\epsilon-\delta, 0.2+2\epsilon-\delta, \cdots, 0.2+2n\epsilon-\delta$.
$\mathcal{B}_n$ is defined as follows: Place a's at all points as in $\mathcal{A}_n$, except the point $0.2+n\epsilon-\delta$.

Clearly, $\mathcal{A}_n \models \phi$ and $\mathcal{B}_n \not\models \phi$.

It is also easy to see that the *Duplicator* has a winning strategy here, the description of which we skip here. ☐

5.2 The main problem

Here we shall talk about the exact unsolved problem we wish to solve $MTL^c[\mathbf{U}_I, \mathbf{S}_I]$ and $MTL^c[\mathbf{U}_I, \mathbf{S}]$. Note that given a formula $\phi \in MTL^c[\mathbf{U}_I, \mathbf{S}_B]$, we can rewrite ϕ such that it only uses $\mathbf{U}_I, \mathbf{P}_I$ and $\mathbf{S}$ where the intervals used by $\mathbf{P}_I$ are only precise. The idea behind this conversion is $a\mathbf{S}_{(0,1)}b \equiv \mathbf{P}_{[1,1]}(\mathbf{F}_{(0,1)}b \wedge \mathbf{F}_{[1,1]}(a\mathbf{S}b))$.

So without loss of generality, we assume that the formula ϕ uses only $\mathbf{U}_I, \mathbf{P}_I$ and $\mathbf{S}$. Suppose the modal depth of ϕ is n, then given any two models A and B which are distinguished by ϕ, there is an n-round $MTL^c[\mathbf{U}_I, \mathbf{P}_I, \mathbf{S}]$ EF game on A and B which the *Spoiler* wins. We show that for the same words, but using only an $MTL^c[\mathbf{U}_I, S]$ EF game, the *Spoiler* can win in at most $f(n)$ rounds. Given ϕ, $f(n)$ is a fixed constant, so this fact along with the EF Theorem (Theorem 4) can be used to conclude that the logics are in fact equal.

We believe that we shall soon be able to prove our conjecture that the two logics are equal in the continuous case.

Chapter 6

Future work

We have realized that EF games are very insightful as far as understanding expressiveness of a logic is concerned. An EF game that distinguishes two logics comes up from the exact formula that is in exactly one of the logics. On the other hand, an EF game on two words in which the spoiler wins can be used to construct in a structured way the formula that actually distinguishes the two words. Thus EF games are a very helpful tool in studying timed logics. Nevertheless, there are a few issues with this approach. The most important being that there is no structured way to come up with an EF game that separates two logics. Also, given a game, proving winning strategies for the duplicator might still turn out to be tricky, involving multiple cases. Despite this, the EF games approach seems very promising.
Here are a few directions in which we would like to investigate:

- The MTL hierarchy collapsed in the continuous case, as we saw in Claim 5. But that result made use of punctual intervals to derive the result. So, we would like to find out if a similar hierarchy exists for MITL, or even that collapses.
- We hope to complete the proof of the approach described in Section 5.2 soon.
- The major result of [6] was to show that MTL is a strict subset of TPTL for pointwise semantics. So one important line of thought is whether this result also holds for continuous semantics. This is still an unsolved problem.

Bibliography

[1] Rajeev Alur and Thomas A. Henzinger. A really temporal logic. In *FOCS*, pages 164–169, 1989.

[2] Rajeev Alur and Thomas A. Henzinger. Real-time logics: Complexity and expressiveness. *Inf. Comput.*, 104(1):35–77, 1993.

[3] Kousha Etessami and Thomas Wilke. An until hierarchy for temporal logic. In *LICS*, pages 108–117, 1996.

[4] Orna Lichtenstein and Amir Pnueli. Propositional temporal logics: Decidability and completeness. *Logic Journal of the IGPL*, 8(1):55–85, 2000.

[5] Joël Ouaknine and James Worrell. Some recent results in metric temporal logic. In *FORMATS*, pages 1–13, 2008.

[6] Paritosh K. Pandya and Simoni S. Shah. On expressive powers of timed logics: Comparing boundedness, non-punctuality, and deterministic freezing. In *CONCUR*, pages 60–75, 2011.

[7] Wolfgang Thomas. On the ehrenfeucht-fraïssé game in theoretical computer science. In *TAPSOFT*, pages 559–568, 1993.