The Goemans–Williamson Rounding Alg. for MAXCUT

Just coming up with a feasible solution to a NP-HARD problem is in practice insufficient, as we generally require also a feasible sol. Today, we see a method to guarantee a feasible rounding sol. to MAXCUT that achieves a multiplicative guarantee of \( \approx 0.8789 - \varepsilon \).

Let \( G = (V, E) \) be a graph with \( n = |V|, m = |E| \). Then, we can define MAXCUT as

\[
\text{MAXCUT}(G) = \left\{ \begin{array}{l}
\max \sum_{(i,j) \in E} \frac{1 - x_i x_j}{2} \\
\text{s.t. } x_i \in \mathbb{R} \pm 1 \quad \forall i \in V.
\end{array} \right.
\]

We can consider the SDP relaxation as

\[
\text{SDP}(G) = \left\{ \begin{array}{l}
\sup \sum_{(i,j) \in E} \frac{1 - x_i x_j}{2} \\
\text{s.t. } \|x_i\| = 1 \quad \forall i \in V.
\end{array} \right.
\]

Let’s quickly verify that \( \text{SDP}(G) \geq \text{MAXCUT}(G) \).

**Proof.** Map each vector \( x_i \mapsto \text{a vector } (\pm 1, 0, \ldots, 0) \in \mathbb{R}^n \) and \( x_i x_j \mapsto x_i \cdot x_j \). Then easy to see that \( \text{SDP}(G) \geq \text{MAXCUT}(G) \).

Let’s write the SDP in the canonical form. Make it easier to complete the proof of the correctness of the rounding alg.
Define a matrix \( X = (x_{ij}) \) where \( x_{ij} = x_i \cdot x_j \). The matrix \( X \) is PSD as it can be expressed as

\[
X = \begin{pmatrix}
\frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \\
\frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m}
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_m
\end{pmatrix} = \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_m
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_m
\end{pmatrix}^T
\]

The constraint \( \|x\|_1 = 1 \) is the same as \( \sum_{i} x_i = 1 \) which is equivalent to \( X \cdot E_{ii} = 1 \). As we are maximizing \( \|x\|_1 \), we can simplify to \( X_{ii} \cdot E_{ii} \leq 1 \).

Define \( A = (a_{ij}) \) by \( a_{ij} = \frac{1}{m} \frac{1}{\sqrt{m}} \chi_{(i,j) \in E} \).

\[
\max_{x \in \mathbb{R}^m} \sum_{(i,j) \in E} \frac{1 - x_i x_j}{2} = \max \frac{m}{2} \sum_{i=1}^m (1 - a_{ii} x_i^2) = \frac{m}{2} + \max (-A \cdot X)
\]

\[
\text{SDP}(G) = \begin{cases}
\sup (-A \cdot X) \\
\text{s.t. } E_{ii} \cdot X \leq 1, \quad i = 1, \ldots, n \\
X \geq 0
\end{cases}
\]

Let's take a second look to explore how the SDP can be visualized.

Let \( G = K_3 = \Delta \). Obviously, the MAXCUT \( (K_3) = 2 \). Consider the vector definition of SDP. For vectors \( x_1, x_2, x_3 \in \mathbb{R}^m \) with norm 1, \( x_i \cdot x_j = \cos(\Theta_{ij}) \) where \( \Theta_{ij} = \) the angle between them.

Goal is then to maximize \( 1 - \cos(\Theta_{ij}) \) or minimize \( \cos(\Theta_{ij}) \).

i.e. all are close to \( 180^\circ \) apart.

\[
\cos(\Theta_{ij}) = \frac{1}{2} \Rightarrow \text{all } \Theta_{ij} = 180^\circ.
\]

\[
\text{circle}
\]

\[
\text{ellipse}
\]
Choose \( \theta = \frac{\pi}{3} \).

Then \( \cos(\theta + \frac{\pi}{3}) = -\frac{1}{2} \). So,

\[
\text{SDP}(G) \geq 3 \left( \frac{1 - (-\frac{1}{2})}{2} \right) = \frac{9}{4}.
\]

So \( \frac{\text{SDP}(K_3)}{\text{MAXCUT}(K_3)} = \frac{\frac{9}{4}}{2} = \frac{9}{8} \), i.e., no algorithm should hope for a multiplicative guarantee too strong.

The Rounding Procedure:

Starting from an optimal sol. of \((x_i)\), select a uniformly random \( y \in \mathbb{R}^n \) and apply the rounding procedure

\[
x_i \rightarrow \text{sign}(x_i \cdot y) := \varepsilon_i.
\]

Unlike other problems that are NP-hard, any assignment \( \varepsilon \in \{-1, 1\}^n \) to the vertices is a valid assignment. But, we just need to ensure that the cut produced is close to optimal.

Prop. For any \( i, j \), \( P(y \mid \varepsilon_i \neq \varepsilon_j) = \arccos \left( \frac{x_i \cdot x_j}{\|x_i\| \|x_j\|} \right) \).

Let \( H \) be the half plane \( \{ y \mid y \cdot x_i > 0 \} \) to \( y \).

Split \( S^{n-1} \) into two halves, where \( x \cdot y > 0 \) in one and \( x \cdot y < 0 \) in the other.

Consider \( C \) circle containing both \( x_i \) and \( x_j \) on \( S^{n-1} \).

Then \( H \cap C \) is two anti-podal pts.

If one of the anti-podal pts. is on the arc \( x_i, x_j \), then they are in different halves.
That occurs with probability \( \frac{\Theta}{\pi} \), where \( \Theta \) is the angle between them.

As \( x_i \cdot x_j = \|x_i\| \|x_j\| \cos \Theta \Rightarrow \Theta = \arccos (x_i \cdot x_j) \).

So \( \Pr \left( E_i \neq E_j \right) = \frac{\arccos (x_i \cdot x_j)}{\pi} \).

Let \( F : [-1, 1] \rightarrow [0, 1] \) be the \( F \). \( F'(\sigma) = \frac{\arccos (\sigma)}{\pi} \).

\[
\mathbb{E}(GW(G)) = \sum_{i,j} A_{ij} \frac{\arccos (x_i \cdot x_j)}{\pi} \quad \text{SDP}(G) = \sum_{i,j} A_{ij} \left( \frac{1}{2} - \frac{1}{2} x_i \cdot x_j \right)
\]

How much worse is the rounding than the SDP?

We can verify the largest gap occurs at approximately

\(-0.689\), and the ratio is approx. \(0.879\).

Therefore, since we know that

\(GW(G) \geq 0.879 \text{ SDP}(G)\).

Recall that \( \text{SDP}(G) \geq \text{MAXCUT}(G)\). And as \( GW \) defines a cut, then

\(\text{MAXCUT}(G) \geq \otimes GW(G) \Rightarrow 0.879 \text{ SDP}(G) \leq GW(G) \leq \text{MAXCUT}(G) \leq \text{SDP}(G)\).

So, \( GW \) also yields a \(0.879\) approx. of MAXCUT.

* The \(0.879\)-approx. result comes from the inability to calculate a \( y \in \{-1, 1\}^n \) perfectly randomly. The best strategy done is to choose \( y \sim [N(0, 1)]^n\).
But, we're not done yet. The argument here was made for the worst case, but if the SDP is close to optimal, then we can actually make stronger arguments about the GW ranking.

Consider the tangent convex envelope of $B$, define it $\tilde{B}$. i.e. we need to choose the pt. $c_1$ s.t. from $c_1$ the tangent curve is always below the graph of $B$. To solve, find the $c_1$ s.t. the tangent from $c_1$ to $(1,0)$.

Solve: $O = \tilde{B}(1) = B(c_1) + B(c_1)(\sigma - c_1)$

\[ O = \frac{\arccos(c_1)}{\pi} - \frac{(1 - c_1)}{\pi \sqrt{1 - c_1^2}}. \]

i.e. $\arccos(c_1) = \frac{1 - c_1}{\sqrt{1 - c_1^2}}$.

Assume $c_1 = \cos \theta$ for some $\theta \in (0, \pi)$. Therefore,

\[ \theta = \frac{1 - \cos \theta}{\sin \theta} = \frac{(\sin \theta)(1 - \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)} = \frac{\sin \theta}{1 + \cos \theta} : = \tan \frac{\theta}{2}. \]

So $\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$.

So, $c_1 = \cos \theta$ and if we define $c_2 = \frac{2}{\pi \sin \theta}$, where $\theta = \tan \frac{\theta}{2}$. Then,

\[ B(\sigma) = \begin{cases} B(\sigma) & \text{if } \sigma < c_1, \\ B(c_1) - \frac{c_1}{2}(\sigma - c_1) & \text{o.w.} \end{cases} \]

\[ c_1 \sim 0.689, \quad c_2 \sim 0.879. \]
Prop. If \( \text{SDP}(G) \geq \delta m \) \( \Rightarrow \) \( \text{E}(GW(G)) \geq \beta (1 - 2\delta) m \).

\[ \frac{m}{2} - \sum_{(i,j) \in E} x_i \cdot x_j \geq \delta m \quad \text{where} \quad (\mathcal{X}) \quad \text{is optimal SDP sol.} \]

\[ \Rightarrow \sum_{(i,j) \in E} x_i \cdot x_j \leq (1 - 2\delta) m. \]

We have \( \text{Pr}_y (e_i \neq e_j) = \beta (x_i \cdot x_j) \geq \beta (x_i \cdot x_j) \). As \( \beta \) is convex,

\[ \sum \beta (y_i) \geq \beta \left( \sum y_i \right). \]

And as \( \beta \) is a negative \( \beta \), \( \beta (y) \geq \beta (x) \) \( \forall \ y < x \).

\[ \Rightarrow \text{E}(GW(G)) = \sum_{(i,j) \in E} \text{Pr}_y (e_i \neq e_j) \]

\[ \geq \sum_{(i,j) \in E} \beta (x_i \cdot x_j) \]

\[ \geq \beta \left( \sum_{(i,j) \in E} x_i \cdot x_j \right) \quad \text{convexity} \]

\[ \geq \beta (1 - 2\delta) m \quad \text{negativity} \]

\[ \geq \beta (1 - 2\delta) m \quad \text{convexity} \]

Now let's assume \( \text{SDP}(G) \geq (1 - \varepsilon) m \) for small \( \varepsilon > 0 \). Then, we will see that

\[ \text{E}(GW(G)) \geq \left( 1 - \frac{2}{\pi} \sqrt{\varepsilon} - o(\sqrt{\varepsilon}) \right) m. \]

\[ \text{Pr} \quad \text{By previous,} \quad \text{E}(GW(G)) \geq \beta (1 - 2\delta) m = \beta (-1 + 2\varepsilon) m \quad \text{where} \quad 1 - \varepsilon = \delta. \]

Then for \( 0 < \varepsilon \leq \frac{1 + \alpha}{2} \), \( \beta (-1 + 2\varepsilon) = \beta (-1 + 2\varepsilon) = \frac{\arccos (-1 + 2\varepsilon)}{\pi}. \)
Now, consider angle $\gamma$ s.t. $\cos(\gamma) = 1 + 2\varepsilon$. Then $\beta \left( 1 + 2\varepsilon \right) = \frac{\gamma}{\pi}$.

Or, $\cos(\gamma - \pi) = \cos(\pi - \gamma) = 1 - 2\varepsilon$. Note $\gamma \approx \pi$, so for $\varphi := \pi - \gamma$.

Small angle approximation, $1 - 2\varepsilon = 1 - \frac{\varphi^2}{2} = o(\varphi^2)$.

A little more and, $\varphi = 2\sqrt{\varepsilon} + o(\sqrt{\varepsilon})$.

So, $\gamma = \pi - 2\sqrt{\varepsilon} - o(\sqrt{\varepsilon})$. Thus,

\[
E(GW(G)) \geq \frac{\pi - 2\sqrt{\varepsilon} - o(\sqrt{\varepsilon})}{\pi} m - \left( 1 - \frac{2}{\pi} \sqrt{\varepsilon} - o(\sqrt{\varepsilon}) \right) m.
\]