

# Computing Stackelberg Equilibria of Large General-Sum Games

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**Abstract.** We study the computational complexity of finding Stackelberg Equilibria in general-sum games, where the set of pure strategies of the leader and the followers are exponentially large in a natural representation of the problem.

In *zero-sum* games, the notion of a Stackelberg equilibrium coincides with the notion of a *Nash Equilibrium* [Korzhyk et al., 2011b]. Finding these equilibrium concepts in zero-sum games can be efficiently done when the players have polynomially many pure strategies or when (in addition to some structural properties) a best-response oracle is available [Ahmadinejad et al., 2016, Dudík et al., 2017, Kalai and Vempala, 2005]. Despite such advancements in the case of zero-sum games, little is known for general-sum games.

In light of the above, we examine the computational complexity of computing a Stackelberg equilibrium in large general-sum games. We show that while there are natural large general-sum games where the Stackelberg Equilibria can be computed efficiently if the Nash equilibrium in its zero-sum form could be computed efficiently, in general, structural properties that allow for efficient computation of Nash equilibrium in zero-sum games are not sufficient for computing Stackelberg equilibria in general-sum games.

## 1 Introduction

Recent years have witnessed significant interest in Stackelberg games and their equilibria. A *Stackelberg game* models an interaction between two players, a *leader* and a *follower*, where the leader’s goal is to *commit* to a randomized strategy that yields the highest utility, given that the follower responds by choosing an action that is best for itself. Such a pair of strategies is called a *Stackelberg equilibrium (SE)*. The interest in these games is driven, in part, by their applications to security [Tambe, 2011] and their adoption by major security agencies such as the US Coast Guard, the Federal Air Marshals Service, and the Los Angeles Airport Police.

Standard approaches for finding a Stackelberg equilibrium, such as the Multiple LPs approach of Conitzer and Sandholm [2006], run in time polynomial in the number of pure strategies of the leader and follower. As Stackelberg games and their applications have become more prevalent, they are increasingly used to model complex scenarios where one or both players’ strategy sets are *exponentially large in a natural representation of the problem*, in which case existing approaches are not computationally feasible.

In this work, we consider such “large” games and ask whether there are computationally efficient algorithms for finding their Stackelberg equilibria.

Of course, such algorithms cannot exist without some assumptions on the problem structure. Here, we review the common assumptions and approaches for computing minimax-optimal solutions in large *zero-sum* games, where minimax strategies, Nash equilibria, and Stackelberg equilibria all coincide. Computing these equilibrium concepts in 2-player zero-sum games has received significant attention [Ahmadinejad et al., 2016, Behnezhad et al., 2018a, 2019, 2017a, 2018b, Conitzer and Sandholm, 2006, Dudík et al., 2017, Freund and Schapire, 1995, Garg et al., 2011, Immorlica et al., 2011, Von Neumann and Morgenstern, 1945, Wang and Shroff, 2017, Xu, 2016]. For large zero-sum games, two structural assumptions that have proven useful in computing a Nash equilibrium are the ability to efficiently optimize a linear function over the strategy space of each player [Ahmadinejad et al., 2016] and the ability to compute the best-response of each player against a mixed strategy of the other combined with a decomposibility property of the action set [Dudík et al., 2017].

In general-sum games, however, Stackelberg, Nash, and Minimax equilibria diverge. In general-sum games the leader can benefit from committing to a mixed strategy and obtain a more favorable Stackelberg equilibrium than any Nash equilibrium. From the algorithmic perspective, a Stackelberg equilibrium in a general-sum game can be computed efficiently when the game is small. That is, there are algorithms, such as the Multiple LPs approach of Conitzer and Sandholm [2006], that run in time  $\text{poly}(|S_L|, |S_F|)$  where  $S_L$  and  $S_F$  are the set of pure strategies of the leader and follower, respectively. While this method is an efficient approach for computing a Stackelberg equilibrium in small games, it becomes computationally inefficient in many natural scenarios where the set of actions of the leader or follower is exponential in a natural representation of the game. Examples of such settings include games inspired by applications to security, where either the actions of the leader or the follower represent sets of edges in a graph. As opposed to the zero-sum case for which existence of certain structural properties are known to lead to efficient computation of the equilibrium concepts, computation of Stackelberg equilibrium in large general-sum games has remained mostly unexplored.

## 1.1 Our Results and Contributions

In light of the above, we examine the computational complexity of computing Stackelberg equilibria in large general-sum games. Specifically, we consider two classes of general-sum games, both of which demonstrate structural properties that under the zero-sum assumption would lead to efficient algorithms for computing the minimax optimal strategies. For the first class of games, we give an efficient algorithm for computing a Stackelberg Equilibrium. In the second class of games, we show that even approximating the Stackelberg equilibrium is NP-Hard. This drives home the main message of this work, that is *while there are natural large general-sum games where the Stackelberg Equilibria can be computed efficiently if the Nash equilibrium in its zero-sum form could be computed efficiently, in general, structural properties that allow for efficient computation of Nash equilibrium in zero-sum games are not sufficient for computing Stackelberg equilibria in general-sum games.*

In more details, the two classes of games we work with are as follow.

**INCENTIVE GAMES** In Section 3, we introduce a class of games called **INCENTIVE GAMES**. In these games, the actions of the leader can be described as two-part actions, the first part of the action is an element of a set and the second part of the action is a set of incentives to the follower for playing certain actions.

As a motivating example consider a taxation scenario. In this setting, a government agency (e.g., IRS) takes the role of the leader and a taxpayer is the follower. A number of investments, indicated by the set  $E$ , are available to the taxpayer. Each investment  $e$  has a return of  $c_e$  to the taxpayer. The taxpayer invests in a package of investments  $S \subseteq E$  that has the highest net payoff. The government agency is interested in taxing these investments in order to maximize the tax revenue. To do so, the agency allocates 1 unit of taxes between these investments. There are two types of taxation mechanisms. First is taxing an individual investment  $e$  by some amount  $x_e$ . The second is to provide tax relief  $v_S$  for a package of options the taxpayer has invested in. Examples of the second type of taxation mechanism include United States federal residential renewable energy tax credit that offers a tax break to individuals who have invested in home electric power storage, e.g., batteries, and home-generated renewable energy, e.g., solar panels, but no tax break to those who have invested in the former without the latter [EnergySage, 2019, U.S. Department of Energy, 2019]. The tax revenue and the net payoff the taxpayer respectively receive from individual taxes  $x_e$  and combinatorial tax reliefs  $v_S$  when the taxpayer invests in investments  $S$  are  $\sum_{e \in S} x_e - v_S$  and  $\sum_{e \in S} (-x_e + c_e) + v_S$ . It is not hard to see that these tax breaks play an essential role in the design of tax systems. Not only they increase the total tax revenue obtainable by a tax system (See Example 1 for an illustration) but they can also be used to incentivize the taxpayers to take actions that are more beneficial to the government.

More generally, we consider Stackelberg games and we consider a family of sets  $\mathcal{S} \subseteq \{0, 1\}^E$  and leader and follower element payoffs,  $C_e$  and  $c_e$ , respectively, for all  $e \in E$ . A pure strategy of the leader is to choose  $e \in E$  and a vector of *incentives*  $\mathbf{v} \in [0, 1]^{|\mathcal{S}|}$ , such that  $\|\mathbf{v}\|_0 \in \text{poly}(|E|)$ .<sup>5</sup> A pure strategy of the follower is to choose one set  $S \in \mathcal{S}$ . The payoff of the leader and follower are defined, respectively, by

$$\begin{aligned} U_L((e, \mathbf{v}), S) &= 1_{e \in S} - v_S + C_e, \\ U_F((e, \mathbf{v}), S) &= -1_{e \in S} + v_S + \sum_{e' \in S} c_{e'}, \end{aligned}$$

that is, the players receive non-zero-sum utilities from their individual choices, i.e.,  $C_e$  and  $\sum_{e' \in S} c_{e'}$ , and zero-sum utilities from choosing actions that intersect, i.e.,  $\pm 1_{e \in S}$ , and from the incentives provided on the followers actions sets, i.e.,  $\pm v_S$ .

We first note that when  $c_e$  and  $C_e$  are set to 0 for all  $e \in E$ , this game is zero-sum and can be efficiently solved when each player can compute its best-response to any choice of mixed strategy of the other player, i.e., optimize a linear function over the strategy space of the other player using existing results [Ahmadinejad et al., 2016, Dudík et al., 2017, Kalai and Vempala, 2005].

<sup>5</sup> The sparsity requirement is such that the leader can communicate its strategy to the follower efficiently.

When  $c_e$  and  $C_e$  are non-zero, we show that the leader can obtain a higher payoff equilibrium if it could make additional commitments in the form of incentives for the follower, i.e., can play non-zero  $\mathbf{v}$ . An interesting aspect of this game is that it is derived from a simple Stackelberg game model (where  $\mathbf{v} = \mathbf{0}$ ) by adding zero-sum payoffs that *only benefit the follower*. Yet, the leader’s payoff in the Stackelberg equilibrium of the new game is much higher than its payoff in the original game. Moreover, as we show in Theorem 1 there is a polynomial time algorithm for finding the Stackelberg equilibrium of such games when the leader can optimize a linear function over the actions of the follower, which is a similar condition to the ones used for computing Stackelberg equilibria in large zero-sum games [Ahmadinejad et al., 2016, Dudík et al., 2017, Kalai and Vempala, 2005].

**PERMUTED MATCHING GAME** In Section 4, we introduce a non-zero-sum game called **PERMUTED MATCHING**. In this game, there is a graph  $G = (V, E)$  and a permutation  $\pi : E \rightarrow E$ . The set of pure strategies of the leader and follower is the set of all matchings in  $G$ . The goal of the leader is to maximize the intersection of its matching with the  $\pi$ -transformation of the matching of the follower. On the other hand, the goal of the follower is to maximize the intersection of the two matchings, with no regards to  $\pi$ . More formally, for  $S \subseteq E$  we define  $\pi(S) = \{e \in S | \pi(e)\}$ . Then the utility of the leader and follower are defined, respectively, by

$$U_L(M_1, M_2) = |M_1 \cap \pi(M_2)| \quad U_F(M_1, M_2) = |M_1 \cap M_2|.$$

It is not hard to see that, in this game, the problem of finding a best response for a player reduces to computing maximum weighted matching of  $G$  and can be solved in polynomial time. This would have been sufficient for getting a polynomial time algorithm for finding a Stackelberg equilibrium had the game been a zero-sum [Ahmadinejad et al., 2016, Dudík et al., 2017, Kalai and Vempala, 2005]. In a sharp contrast, however, we show that computing a Stackelberg equilibrium of this general-sum game is APX-hard, even though, we can compute player’s best-response efficiently.

We obtain this hardness result via two reductions. First, we define the following computational problem:

**$\pi$ -TRANSFORMATION-IDENTICAL-MATCHING:** Given a graph  $G$  and a mapping  $\pi : E(G) \rightarrow E(G)$  over the edges of  $G$ , find a matching  $M$  of  $G$  that maximizes  $|M \cap \pi(M)|$ .

We next show that computing an approximate Stackelberg equilibrium of the **PERMUTED MATCHING** game is at least as hard as computing an approximate solution for the  **$\pi$ -TRANSFORMATION-IDENTICAL-MATCHING** problem. The crux of the argument is that if in an instance of the  **$\pi$ -TRANSFORMATION-IDENTICAL-MATCHING** problem there exists a matching which is almost identical to its  $\pi$ -transformation, then a Stackelberg equilibrium of the **PERMUTED MATCHING** game is closely related to that matching. Thus, any solution for the **PERMUTED MATCHING** game can be turned into a solution for  **$\pi$ -TRANSFORMATION-IDENTICAL-MATCHING** with almost the same quality. In the second step, we reduce the  **$\pi$ -TRANSFORMATION-IDENTICAL-MATCHING** problem to the **MAXIMUM 3D MATCHING** problem, which is known to be APX-hard [Petrank, 1994].

We note that our results strengthen the existing hardness results of [Letchford and Conitzer \[2010\]](#), [Li et al. \[2016\]](#) that showed that computing Stackelberg equilibrium is *NP-hard*<sup>6</sup>. Our APX-hardness result shows that one cannot even approximate the Stackelberg equilibria of large games within *an arbitrary constant factor*, even when best-response can be efficiently computed.

## 1.2 Related Work

There is an extensive body of work investigating the complexity of solving Security games, which is a special case of computing Stackelberg equilibria (see *e.g.* [Basilico et al. \[2009\]](#), [Behnezhad et al. \[2017b\]](#), [Letchford and Vorobeychik \[2011\]](#), [Tambe \[2011\]](#), [Xu \[2016\]](#), [Xu et al. \[2014\]](#)).

*zero-sum games* Several algorithms have been proposed for finding the Stackelberg equilibria of a special case of security games called the spatio-temporal security games [[Behnezhad et al., 2017b](#), [Xu et al., 2014](#)]. These games are zero-sum by definition, where Stackelberg equilibria, Nash equilibria, and Minimax equilibria all coincide. In comparison, our work focuses on general-sum games.

*Smaller general-sum games* Several works have introduced polynomial time algorithms for computing Stackelberg equilibria in games where *only one player's* strategy set is exponentially large [[Kiekintveld et al., 2009](#), [Xu, 2016](#)]. A common approach used in this case is the Multiple LPs approach of [Conitzer and Sandholm \[2006\]](#) that runs in  $\text{poly}(|S_L|, |S_F|)$ . In this approach one creates a separate Linear Program for every action  $y \in S_F$  of the follower, where the variables represent the probability assigned to the actions of the leader, the objective maximizes the expected payoff of the leader, and the constraints assure that action  $y$  is the best-response of the follower. This method can be implemented efficiently even when the leader's strategy set is exponentially large, *e.g.*, when a separation oracle can be implemented efficiently. In comparison, our main computational result focus on settings where both the leader and follower have exponentially many strategies.

*Existing hardness results* [Letchford and Conitzer \[2010\]](#) studied the computational complexity of extensive form games and proved a closely related hardness result.

They showed that computing Stackelberg equilibrium of a game is weakly NP-hard using a reduction from Knapsack. Interestingly, one can efficiently compute player best-response in their setting. In comparison, our hardness result improves over these results by showing that Stackelberg equilibria are hard to approximate within arbitrary constant factor even when player best-response can be computed efficiently.

A number of works have investigated the relationship between the Stackelberg equilibria and Nash equilibria of security games and have shown that computing a Stackelberg equilibrium is at least as hard as computing a Nash equilibrium of general-sum

<sup>6</sup> Interestingly, it is not hard to show that player best-response can also be computed efficiently in the games used by [Letchford and Conitzer \[2010\]](#), [Li et al. \[2016\]](#), although this was not central to their results.

games. [Korzhyk et al. \[2011a\]](#) studied a special class of general-sum Stackelberg Security games where any Stackelberg Equilibrium is also a Nash equilibrium. This shows that computing Stackelberg equilibria is harder than computing Nash equilibria. [Li et al. \[2016\]](#) studied Bayesian Stackelberg Games, where there is additional uncertainty about the attacker and show that computing the Stackelberg equilibrium is hard, and introduce an exponential time algorithm for computing the Nash equilibria. In comparison, our work shows that Stackelberg equilibria are hard to approximate even when players best-response is easy to compute. That is, we show a gap between the computational complexity of approximating Stackelberg equilibrium of a general-sum game and that of its corresponding zero-sum variant.

## 2 Preliminaries

Throughout this paper, we study Stackelberg equilibria of large games. Our emphasis is on two player games and therefore we denote the players by L (leader) and F (follower). Let  $S_L$  and  $S_F$  be the set of actions (pure strategies) of players L and F. For a pair of pure strategies  $x \in S_L$  and  $y \in S_F$ , we denote the payoffs of players L and F by  $U_L(x, y)$  and  $U_F(x, y)$ , respectively. Similarly, for a pair of mixed strategies  $X$  and  $Y$  we denote the payoffs by

$$\begin{aligned} U_L(X, Y) &= \mathbb{E}_{x \sim X, y \sim Y}[U_L(x, y)] \\ U_F(X, Y) &= \mathbb{E}_{x \sim X, y \sim Y}[U_F(x, y)]. \end{aligned}$$

In Stackelberg games, the leader commits to a (possibly mixed) strategy  $X$  and plays this strategy. The follower then plays a best response against  $X$ ,  $b(X)$ , according to her payoff function. Since the follower goes second its best-response is a deterministic action  $b(X) = \max_y U_F(X, y)$ . In case there is more than one best response for the follower, we assume she plays the one that maximizes the payoff of the leader. A pair of strategies  $X$  and  $y$  are in Stackelberg equilibrium if  $y$  is a best response of the follower against  $X$  and  $X$  maximizes the payoff of the leader, subject to the follower playing a best response.

## 3 INCENTIVE GAMES

In this section, we discuss a class of Stackelberg games where the leader has the ability to make additional commitments in the form of additional incentives to the follower. Recall that a natural scenario that can be addressed by this Stackelberg model is *taxation*. In this case the leader can set taxes on individual investments but can also provide tax breaks on bundles of investments that the tax payer has invested in. We first show how these additional combinatorial incentives can improve the leader's payoff significantly and then show polynomial time algorithms for computing a Stackelberg equilibrium in this model.

Let us first recall the definition of INCENTIVE GAMES. In this model, we consider a set of elements  $E$ , a family of its subsets  $\mathcal{S} \subseteq \{0, 1\}^E$  and rewards  $C_e$  and  $c_e$  for all  $e \in E$ . The set of pure strategies of the leader is  $S_L = E \times [0, 1]^{|\mathcal{S}|}$ . That is, each action

of the leader has two parts, the first part is an element  $e \in E$  and the second part is a vector of *incentives*  $\mathbf{v} \in [0, 1]^{|S|}$ . We assume that the leader is restricted to playing incentive vectors  $\|\mathbf{v}\|_0 \in \text{poly}(|E|)$ .<sup>7</sup> The follower's strategy set is  $S_F = S$ . The leader and follower payoffs are as follows.

$$U_L((e, \mathbf{v}), S) = 1_{e \in S} - v_S + C_e, \text{ and } U_F((e, \mathbf{v}), S) = -1_{e \in S} + v_S + \sum_{e' \in S} c_{e'}, \quad (1)$$

that is, the players receive non-zero-sum utilities from their individual choices, i.e.,  $C_e$  and  $\sum_{e \in S} c_{e'}$ , and zero-sum utilities from choosing actions that intersect, i.e.,  $\pm 1_{e \in S}$ , and from the incentives provided on the followers actions sets, i.e.,  $\pm v_S$ . For ease of exposition and by the linearity of the payoffs, we denote a mixed strategy of the leader by  $(\mathbf{x}, \mathbf{V})$ , where  $x_e$  is the probability with which the first part of the leader's action is  $e$  and  $V_S$  is the expected incentive provided on action  $S$  in the second part of the leader's action. Note that in this case, the expected utilities of the leader and follower are

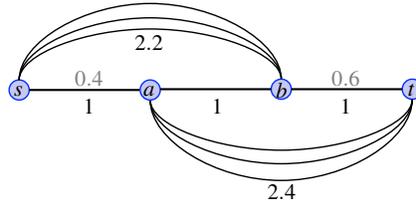
$$U_L((\mathbf{x}, \mathbf{V}), S) = \sum_{e \in S} x_e - V_S + \sum_{e \in E} x_e C_e \quad (2)$$

$$U_F((\mathbf{x}, \mathbf{V}), S) = \sum_{e \in S} (-x_e + c_e) + V_S. \quad (3)$$

Let us first consider a variation of INCENTIVE GAMES where the leader cannot provide additional incentives to the follower, i.e.,  $S_L = E \times \mathbf{0}$ . The only difference between these games is that INCENTIVE GAMES are amended by allowing zero-sum non-negative payments  $\mathbf{v}$  that benefit the follower solely. One might wonder if the commitment to make additional payments  $\mathbf{v}$  to the follower can ever be beneficial to the leader. This is exactly what we demonstrate in the next example. That is, by allowing the leader to make additional zero-sum payoffs that only benefit the follower, we can obtain Stackelberg equilibria that have much higher payoff to the leader.

*Example 1.* Consider a graph instance in Figure 1,  $E$  is the set of all edges,  $S$  is the set of all  $s$ - $t$  paths, there are no edge payoff to the leader, i.e.,  $C_e = 0$  for all  $e \in E$ , and the edge payoff to the follower,  $c_e$ , are the *negative* of the edges costs that are denoted below each edge. That is, this is an instance where the follower is responding by choosing a shortest path with respect to the edge weights that correspond to the probability with which the leader plays them. Note that, since there are many parallel edges  $sb$  and  $at$ , the leaders optimal strategy (with or without additional commitment) is only supported on edges,  $sa$ ,  $ab$ , and  $bt$ . It is not hard to see that without any additional commitment, the Stackelberg equilibrium involves the leader playing edge  $sa$  with probability  $x_{sa} = 0.4$  and edge  $bt$  with probability  $x_{bt} = 0.6$ , and all other edges with probability  $x_e = 0$ . Note that in such a mixed strategy the follower chooses path  $a, b, t$  and the leader's payoff is 0.6. On the other hand, when the leader commits to providing additional incentive (or discount in the cost of a path) of  $v_{sabt} = 0.2$  on path  $s, a, b, t$ , the follower best responds by choosing path  $s, a, b, t$  and the leader's payoff is 0.8.

<sup>7</sup> The sparsity requirement is such that the leader can communicate its strategy to the follower efficiently.



**Fig. 1.** An example where additional commitment increases the leader’s payoff in the Stackelberg equilibrium. The follower’s cost for each edge is denoted *below* the edge, i.e.,  $-c_{sa} = -c_{ab} = -c_{bt} = 1$  and  $-c_{ab} = 2.2$  and  $-c_{at} = 2.4$  for all the edges between  $s$  and  $b$ , and between  $a$  and  $t$ . The mixed strategy of the leader is denoted in gray *above* the edges.

Our main theorem in this section show that there is a polynomial time algorithm for finding the Stackelberg equilibrium of this modified game.

**Theorem 1.** *There is a polynomial time algorithm for finding a Stackelberg equilibrium if one can solve the following problem in polynomial time: Given  $\mathbf{x}$  and value  $W$ , return  $S \in \mathcal{S}$ , such that  $\sum_{e \in S} (-x_e + c_e) \leq -W$ , or return “None” if no such  $S \in \mathcal{S}$  exists.*<sup>8</sup>

At a high level, we show that a Stackelberg equilibrium,  $(\mathbf{x}^*, \mathbf{V}^*)$ , can be found by finding the optimal solution  $(\mathbf{x}, \mathbf{0})$  (with no additional incentives) that involves maximizing the followers payoff of the best response, and then providing enough incentive on one of the follower’s actions. In particular, we choose to provide incentive on the specific  $S \in \mathcal{S}$  that constitutes the best response of the follower to the mixed strategy  $(\mathbf{0}, \mathbf{0})$ .

For the first step of this proof, we consider the following LP, which can be efficiently solved by the separation oracle given in Theorem 1,

$$\begin{aligned} \max_{\mathbf{x}, W} \quad & W + \sum_{e \in E} x_e c_e \\ \forall S \in \mathcal{S}, \quad & \sum_{e \in S} (-x_e + c_e) \leq -W. \end{aligned} \tag{4}$$

Let  $\mathbf{x}^*, W^*$  be the solution to the above LP. Furthermore, let  $S^* = \arg \max_{S \in \mathcal{S}} \sum_{e \in S} c_e$ , and consider the incentive commitments  $V_{S^*}^* = -W^* - \sum_{e \in S^*} (-x_e^* + c_e)$ , and  $V_0^* = 0$  for all  $S \neq S^*$ . That is, we provide enough incentive on set  $S^*$  such that it becomes the best response for the follower.

To prove Theorem 1, we first prove a lower bound on the incentive needed to make an action the best response of the follower.

**Lemma 1.** *Let  $(\mathbf{x}', \mathbf{V}')$  be any mixed strategy of the leader and let  $S' = b(\mathbf{x}', \mathbf{V}')$  be the corresponding best response of the follower. Let  $W' = -\max_{S \in \mathcal{S}} \sum_{e \in S} (-x'_e + c_e)$ . We have,*

$$V_{S'}' \geq -W' - \sum_{e \in S'} (-x'_e + c_e).$$

<sup>8</sup> An example of a game where this linear program can be solved efficiently is the shortest path game in Example 1.

In the interest of space, we skip the proof of Lemma 1 here. The reader may find a complete proof in the full-version of the paper.

*Proof (of Theorem 1).* Let  $(\mathbf{x}^*, W^*)$  be the solution to Equation 4. Let  $S^* = \arg \max_{S \in \mathcal{S}} \sum_{e \in S} c_e$ , and let  $V_{S^*}^* = -W^* - \sum_{e \in S^*} (-x_e^* + c_e)$ , and  $V_S^* = 0$  for all  $S \neq S^*$ . It is clear that  $b(\mathbf{x}^*, \mathbf{V}^*) = S^*$ . Here, we show that  $(\mathbf{x}^*, \mathbf{V}^*)$  is indeed the optimal leader strategy.

For any leader strategy  $(\mathbf{x}', \mathbf{V}')$ , let  $S' = b(\mathbf{x}', \mathbf{V}')$  be the follower's best response. Moreover, let  $W' = -\max_{S \in \mathcal{S}} \sum_{e \in S} (-x'_e + c_e)$ . We have

$$U_L((\mathbf{x}^*, \mathbf{V}^*), S^*) = \sum_{e \in S^*} x_e^* + \sum_{e \in E} x_e^* C_e^* - V_{S^*}^* \quad (5)$$

$$= \sum_{e \in S^*} x_e^* + \sum_{e \in E} x_e^* C_e^* + W^* + \sum_{e \in S^*} (-x_e^* + c_e) \quad (6)$$

$$= \sum_{e \in E} x_e^* C_e^* + W^* + \sum_{e \in S^*} c_e \quad (7)$$

$$\geq \sum_{e \in E} x'_e C'_e + W' + \sum_{e \in S'} c_e, \quad (8)$$

where the second equation is by the definition of  $V_{S^*}^*$  and the last inequality follows by the fact that  $(\mathbf{x}', \mathbf{V}')$  form a valid solution for the LP in Equation 4, for which  $(\mathbf{x}^*, W^*)$  is the optimal solution and the fact that  $S^*$  is chosen to maximize  $\sum_{e \in S^*} c_e$ .

Using Lemma 1 on the value of  $V_{S'}^*$ , we have

$$U_L((\mathbf{x}', \mathbf{V}'), S') = \sum_{e \in S'} x'_e + \sum_{e \in E} x'_e C'_e - V_{S'}^* \quad (9)$$

$$\leq \sum_{e \in S'} x'_e + \sum_{e \in E} x'_e C'_e + W' + \sum_{e \in S'} (-x'_e + c_e) \quad (10)$$

$$= \sum_{e \in E} x'_e C'_e + W' + \sum_{e \in S'} c_e. \quad (11)$$

Equations 8 and 11 complete the proof.

## 4 The PERMUTED MATCHING Game

In this section, we introduce a large but structured general-sum Stackelberg game, called PERMUTED MATCHING, and examine the computational complexity of computing its Stackelberg equilibrium. We show two sets of results for this game. In Section 4.1, we show that this problem is APX-hard. This implies that unlike zero-sum games, finding a Stackelberg equilibrium is computationally hard even if best-response oracles are provided.

The PERMUTED MATCHING game is defined as follows. Consider the leader and follower, L and F. Consider a multigraph  $G = (V, E)$  and a one-to-one mapping (permutation)  $\pi : E \rightarrow E$ . Note that  $\pi$  may take different values on parallel edges of a multi-graph. In the remainder of this section, we refer to a multi-graph  $G$  as a *graph*. In PERMUTED MATCHING, the set of pure strategies of both players is the set of all matchings in  $G$ . Given matchings  $M_L$  and  $M_F$  played by the leader and follower, respectively, we define

$$U_L(M_L, M_F) = |M_L \cap \pi(M_F)|, \text{ and } U_F(M_L, M_F) = |M_L \cap M_F|,$$

where for a set  $S \subseteq E$ , we define  $\pi(S) = \{e \in S \mid \pi(e)\}$ . Note that  $G$  and  $\pi$  are fixed and known to both players.

Let us highlight an important aspect of our hardness result in advance. As the next observation shows, the strategy space of the players in PERMUTED MATCHING, though large, is very structured. At a high level, the reward of each player is a linear function of the action of the other and each player can efficiently optimize a linear function over the strategy space of the other player, for example, each player can compute a best-response to a mixed strategy of the other.

**Observation 1** *There is a polynomial time algorithm such that for every vector  $\mathbf{w} \in [0, 1]^{|E|}$  finds a strategy of the players whose corresponding representation vector  $\mathbf{v}$  maximizes  $\mathbf{v} \cdot \mathbf{w}$ .*

*Proof (Sketch).* This problem reduces to computing a maximum weighted matching of a graph with edge weights  $w_e$  for all  $e \in E$ , which can be performed efficiently [Cormen \[2009\]](#).

In a zero-sum game, existence of such a structure leads to efficient algorithms for computing the Nash or Stackelberg equilibria [Ahmadinejad et al. \[2016\]](#), [Dudík et al. \[2017\]](#), [Kalai and Vempala \[2005\]](#). On the other hand, our APX-hardness result for the PERMUTED MATCHING game shows that existence of this structure does not necessarily lead to efficient algorithms for computing Stackelberg equilibria in general-sum games. With this in mind, we present our hardness results next.

#### 4.1 Hardness of approximation

In this section, we show that it is impossible to approximate a Stackelberg equilibrium of the PERMUTED MATCHING game in polynomial time within an arbitrarily small constant factor unless  $P=NP$ .

Before we proceed to the proof, we define an auxiliary problem and show a hardness result for this problem. Then, we take advantage of this hardness result and show that computing a Stackelberg equilibrium of PERMUTED MATCHING is APX-hard. We call the intermediary problem  $\pi$ -TRANSFORMATION-IDENTICAL-MATCHING and define it as follows: In the input comes an unweighted undirected graph  $G = (V, E)$  and a permutation  $\pi : E \rightarrow E$ . The goal is to find a matching  $M$  of  $G$  that maximizes  $|M \cap \pi(M)|$ . For an instance  $I$  of  $\pi$ -TRANSFORMATION-IDENTICAL-MATCHING, we denote by  $\text{Opt}(I)$  the optimal solution to  $I$  and refer to the value of this solution by  $\text{Val}(I)$ .

We show that  $\pi$ -TRANSFORMATION-IDENTICAL-MATCHING has a *hard gap* at *gap location* 1. That is, it is NP-hard to decide whether for a given graph  $G$  with  $n$  vertices and a function  $\pi$ , the solution of the  $\pi$ -TRANSFORMATION-IDENTICAL-MATCHING problem is exactly equal to  $n/2$  or at most  $(1 - \epsilon)n/2$  for some  $\epsilon > 0$ .

**Lemma 2.** *There exists an  $\epsilon > 0$  such that it is NP-hard to decide whether the solution of the  $\pi$ -TRANSFORMATION-IDENTICAL-MATCHING problem is exactly equal to  $n/2$  or less than  $(1 - \epsilon)n/2$  where  $n$  is the number of the vertices of the input graph.*

In the interest of space, we defer the proof of 2 to the full-version of the paper and just show how this lemma can be used to prove the main result of this section.

**Theorem 2.** *Computing a Stackelberg equilibrium of PERMUTED MATCHING is APX-hard.*

*Proof.* More generally, we show that approximating a Stackelberg equilibrium of the PERMUTED MATCHING game has a hard gap at gap location 1. This immediately implies a hardness of approximation. We show this by a reduction from the  $\pi$ -TRANSFORMATION-IDENTICAL-MATCHING problem. Suppose we are given an instance  $l = \langle G, \pi \rangle$  of the  $\pi$ -TRANSFORMATION-IDENTICAL-MATCHING problem and wish to decide for some  $\varepsilon' > 0$ , whether the solution of this problem achieves a value that is exactly  $n/2$  or is bounded above by  $(1 - \varepsilon')n/2$  where  $n$  is the size of  $G$ . Based on  $l$ , we construct an instance  $\text{Cor}(l)$  of the PERMUTED MATCHING game with the same graph  $G$  and permutation  $\pi$  and seek to find a Stackelberg equilibrium in this game. Note that by definition,  $\text{Val}(l)$  is equal to  $n/2$  if and only if  $G$  contains a perfect matching that is identical to its  $\pi$ -transformation. Otherwise,  $\text{Val}(l)$  is at most  $(1 - \varepsilon')n/2$  and thus any matching of  $G$  shares no more than  $(1 - \varepsilon')n/2$  edges with its  $\pi$ -transformation.

Since for small enough  $\varepsilon'$ , it is NP-hard to distinguish the two cases (Lemma 2), we show that it is NP-hard to approximate a Stackelberg equilibrium of the leader in  $\text{Cor}(l)$ . If  $\text{Val}(l) = n/2$ , then there exists a perfect matching in  $G$  that is identical to its  $\pi$ -transformation. Thus, if both players play this matching in  $\text{Cor}(l)$ , they both get a payoff of  $n/2$ . Notice that  $n/2$  is the maximum possible payoff for any player in this game, therefore, such a strategy pair is a Stackelberg equilibrium. Hence, in case  $\text{Val}(l) = n/2$ , the leader achieves a payoff of  $n/2$  in a Stackelberg equilibrium of the corresponding PERMUTED MATCHING game.

Now, suppose for  $\varepsilon < \varepsilon'/13$  we have a  $1 - \varepsilon$  approximation solution for  $\text{Cor}(l)$ . If  $\text{Val}(l) = n/2$ , then the payoff of the leader in an exact solution of  $\text{Cor}(l)$  is  $n/2$  and therefore a  $1 - \varepsilon$  approximation solution guarantees a payoff of at least  $n(1 - \varepsilon)/2$  for the leader. Let the strategies of the leader and follower be  $X$  and  $y$  in such a solution. Therefore,  $U_L(X, y) \geq n(1 - \varepsilon)/2$ . Notice that  $X$  may be a mixed strategy, but we can assume w.l.g that  $y$  is a pure strategy since there always exists a best response for the follower which is pure. Also, let  $y^*$  be the  $\pi$ -transformation of strategy  $y$ . Let for two matchings  $x$  and  $y$ ,  $\text{common}(x, y)$  denote the number of edges that  $x$  and  $y$  have in common and define  $\text{dist}(x, y) = |x| + |y| - 2\text{common}(x, y)$ . Recall that the payoff of the leader in this game can be formulated as  $\mathbb{E}_{x \sim X}[\text{common}(x, y^*)]$ . Since this value is at least  $n(1 - \varepsilon)/2$  we have:

$$\mathbb{E}_{x \sim X}[\text{common}(x, y^*)] = U_L(X, y) \geq n(1 - \varepsilon)/2$$

and thus

$$\begin{aligned} \mathbb{E}_{x \sim X}[\text{dist}(x, y^*)] &= \mathbb{E}_{x \sim X}[|x| + |y^*| - 2\text{common}(x, y^*)] \\ &\leq \mathbb{E}_{x \sim X}[n - 2\text{common}(x, y^*)] \\ &= n - 2\mathbb{E}_{x \sim X}[\text{common}(x, y^*)] \\ &\leq n - 2n(1 - \varepsilon)/2 \\ &= n\varepsilon. \end{aligned} \tag{12}$$

Inequality (12) shows that  $y^*$  is very similar (in expectation) to a random matching drawn from strategy  $X$ . This intuitively implies that pure strategies of  $X$  should have a

considerable amount of edges in common. It follows from the definition that for three matchings  $x$ ,  $y$ , and  $z$  we have  $\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z)$ . Therefore, we have

$$\begin{aligned}
\mathbb{E}_{x \sim X, x' \sim X}[\text{dist}(x, x')] &\leq \mathbb{E}_{x \sim X, x' \sim X}[\text{dist}(x, y^*) + \text{dist}(y^*, x')] \\
&= \mathbb{E}_{x \sim X, x' \sim X}[\text{dist}(x, y^*) + \text{dist}(x', y^*)] \\
&= \mathbb{E}_{x \sim X}[\text{dist}(x, y^*)] + \mathbb{E}_{x' \sim X}[\text{dist}(x', y^*)] \\
&= 2\mathbb{E}_{x \sim X}[\text{dist}(x, y^*)] \\
&\leq 2\epsilon n
\end{aligned} \tag{13}$$

Recall that the payoff of the follower is determined by the number of edges his matching shares with that of the leader. Moreover, since  $U_L(X, y) \geq n(1 - \epsilon)/2$ , this implies that  $\mathbb{E}_{x \sim X}[|x|] \geq n(1 - \epsilon)/2$ . What Inequality (13) implies is that if the follower plays  $X$  instead of  $y$ , he gets a payoff of at least  $\mathbb{E}_{x \sim X}|x| - 2\epsilon n \geq (1 - 5\epsilon)n/2$  against  $X$ . In other words  $U_F(X, X) \geq (1 - 5\epsilon)n/2$ . Since  $y$  is a best response of the follower against the leader's strategy, we have  $U_F(X, y) \geq U_F(X, X) \geq (1 - 5\epsilon)n/2$  and thus

$$\begin{aligned}
\mathbb{E}_{x \sim X}[\text{common}(x, y)] &= U_F(X, y) \\
&\geq U_F(X, X) \\
&= \mathbb{E}_{x \sim X, x' \sim X}[\text{common}(x, x')] \\
&\geq (1 - 5\epsilon)n/2.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}_{x \sim X}[\text{dist}(x, y)] &= \mathbb{E}_{x \sim X}[|x| + |y| - 2\text{common}(x, y)] \\
&\leq \mathbb{E}_{x \sim X}[n - 2\text{common}(x, y)] \\
&= n - 2\mathbb{E}_{x \sim X}[\text{common}(x, y)] \\
&\leq n - 2(1 - 5\epsilon)n/2 \\
&\leq 5\epsilon n.
\end{aligned} \tag{14}$$

Combining Inequalities (12) and (14) yields

$$\text{dist}(y, y^*) \leq \mathbb{E}_{x \sim X}[\text{dist}(x, y)] + \mathbb{E}_{x \sim X}[\text{dist}(x, y^*)] \leq 6\epsilon n.$$

Therefore, we have  $\text{common}(y, y^*) \geq |y^*| - 6\epsilon n$  and since  $|y^*| \geq (1 - \epsilon)n/2$  we have  $\text{common}(y, y^*) \geq (1 - 13\epsilon)n/2 > (1 - \epsilon')n/2$ . If  $\text{Val}(l) \neq n/2$ , then  $\text{Val}(l)$  is bounded by  $(1 - \epsilon')n/2$ . Therefore,  $\text{common}(y, y^*) > (1 - \epsilon')n/2$  holds if and only if  $\text{Val}(l) = n/2$ . Thus, an approximation solution for  $\text{Cor}(l)$  within a factor  $(1 - \epsilon) > (1 - \epsilon'/13)$  can be used to decide if the solution of  $l$  is  $n/2$  or bounded by  $(1 - \epsilon')n/2$ . This implies a hard gap for the  $\pi$ -TRANSFORMATION-IDENTICAL-MATCHING problem at gap location 1.

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