Satisfiability For Arithmetic Using Simplex

Here a set of linear inequalities

\[ a'_i + \overline{a}_i \cdot x \geq 0 \]

\[ \hat{a}_i' + \overline{a}_i \cdot \hat{x} \geq 0 \]

Equalities, and strict inequalities can be converted to this form for integer variables and coefficients.

Simplex is a linear programming algorithm

- can be used to maximize (minimize) a linear expression subject to a set of linear inequality constraints
- this is useful for satisfiability

To find if \( S \Rightarrow a' + \overline{a} \cdot x \leq 0 \)

first maximize \( a' + \overline{a} \cdot x \) to \( k \) over \( S \)

Means: \( S \Rightarrow a' + \overline{a} \cdot x \leq k \) and \( S \land a' + \overline{a} \cdot x \) is satisfiable

Then \( S \Rightarrow a' + \overline{a} \cdot x \leq 0 \) if \( k \leq 0 \)

- But we need more
  - undo support \[ \rightarrow \] inexpensive
  - generate equalities \[ \rightarrow \] simple
  - generate proofs \[ \rightarrow \] simple

(1)
Representing the inequalities in the Simplex tableau

- For each inequality \( a' + \bar{a} \cdot \bar{x} \geq 0 \) we introduce a slack variable \( s \) and two constraints

\[
\begin{align*}
    s &= a' + \bar{a} \cdot \bar{x} \\
    s &\geq 0
\end{align*}
\]

- We have equality constraints and \( s \geq 0 \) inequality constraints

- We represent them in a Simplex tableau

\[
\begin{array}{cccc}
    C_1 & \ldots & C_j & \ldots & C_c \\
    \hline
    R_1 & T_{i_0} & T_{i_1} & \ldots & T_{i_j} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    R_i & T_{i_0} & \ldots & T_{i_j} & \ldots & T_{i_c} \\
    R_r & T_{r_0} & \ldots & T_{r_j} & \ldots & T_{r_c}
\end{array}
\]

- \( R_i \) are the expressions that own row \( R_i \)
- \( C_j \) - they operate as slack variables
Invariant 1

\[ R_i = T_{i0} + \sum_j T_{ij} \cdot C_j \]

where \( E_1 \sim E_2 \) means that \( E_1 - E_2 \) can be simplified to 0 using the axioms of \((\mathbb{Z}, 0, +)\) as a commutative group.

E.g.,

\[ 2x + y \sim 2(x + 2(y - z)) + 2z - 3y \]

So far we represented only the equality constraints.

To represent the \( \geq 0 \) constraints we mark the owner of a row or a column as "+ restricted" row or column.

Invariant 2

If row \( i \) is + restricted then there is a proof of \( R_i \geq 0 \). Call that \( \text{Proof}(R(i)) \).

Similar for + restricted columns.
Example

\[ x + y \geq -3 \]
\[ x + 1 \geq y \]
\[ -x \geq 4 \]

\[
\begin{array}{ccc}
| & x & y \\
| a^* & 1 & 1 & -1 \\
b & 3 & 1 & 1 \\
c & -4 & -1 & 0 \\
\end{array}
\]

\[ C_1 = x \quad C_2 = y \]
\[ R_1 = x - y + 1 \]
\[ R_2 = x + y + 3 \]
\[ R_3 = -x - 4 \]

But these equations are not satisfiable

\[(x+1\geq y) + (x+y\geq -3) + 2 \cdot (-x\geq 4)\]

\[ = 1 \geq -3 + 8 \]

Idea: a tableau can become unsatisfiable only when restricting rows or columns.

- convenient to work with satisfiable tableaux
- only restrict the tableau if we can verify that it stays satisfiable
Definition

- The simple point of the tableau is obtained by setting all $C_j = 0$ and all $R_i = T_{io}$
  - this satisfies all equality constraints
- A tableau is feasible if the simple point satisfies all row restrictions
  for all $+\text{-rest. row } i \quad T_{io} > 0$

Back to our example

we hold on to the restriction $c \geq 0$

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^+$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$b^+$</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$c$</td>
<td>-4</td>
<td>-1</td>
</tr>
</tbody>
</table>

- this is a feasible tableau
- it would become unfeasible if we add $c \geq 0$
A feasible tableau denotes a convex polyhedron in $\mathbb{R}^r$ such that $\overline{0} \in S$, the polyhedron

- in our case $r=2 \Rightarrow 2D$ polygons

Let $\eta = x - y + 1 \geq 0$

- this sample point satisfies $\eta \geq 0$
- the coordinates of this point $b=0, a=0$
- $b = x + y + 3 \geq 0$

Since the sample point does not satisfy $c \geq 0$ ($-x - 4 \geq 0$), we try to move the sample point to increase the value of $-x - 4$ (currently is $-4$)

**Simplex idea**
- this can be done by translating the sample point along one axis until it hits an enclosing hyperplane
- this step can be repeated a finite number of times
Definition

- Moving the sample point along axis $C_j$ until it reaches the hyperplane corresponding to $R_i = 0$ is called pivoting on $i, j$.

- Pivoting is a Gaussian elimination where $C_j$ is expressed in terms of $R_i$ and placed in all equations.

- Then $R_i$ becomes a column and the sample point is $R_i = 0$.

$$C_k = 0 \quad k \neq j$$

<table>
<thead>
<tr>
<th>$R_i$</th>
<th>$T_{io} - \frac{T_{io} \cdot T_{iv}}{T_{uv}}$</th>
<th>$T_{ij} - \frac{T_{ij} \cdot T_{iv}}{T_{uv}}$</th>
<th>$\frac{T_{iv}}{T_{uv}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{(j)}$</td>
<td>$\downarrow$</td>
<td>$R(u)$</td>
<td></td>
</tr>
</tbody>
</table>

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<th>$R_i$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$C_{(v)}$</td>
<td>$\downarrow$</td>
<td>$R(u)$</td>
<td></td>
</tr>
</tbody>
</table>

$$\frac{T_{io}}{T_{uv}} - \frac{T_{ij}}{T_{uv}} - \frac{1}{T_{uv}}$$
Purpose of pivoting:
- We choose pivot \((u, v)\) such that
  - the new sample point is still feasible
  - the sample value of row \(i\) grows
    (because we start with \(T_{i0} < 0\) and we want to restrict \(R_{i} \geq 0\))

Note that it only makes sense to choose a pivot that \(R_{u} \geq 0 \Rightarrow T_{u0} \geq 0\)

Consider the situation when:
- we try to increase the sample value of \(R_{i}\)
- all entries in row \(i\) are negative and in \(+\)-restricted columns.
  - since \(CC_{i}\) is \(+\)-restricted, \(\Rightarrow \frac{T_{u0}}{T_{uv}} < 0\)

Since \(T_{i0} < 0 \Rightarrow \frac{T_{u0} \cdot T_{i0}}{T_{uv}} > 0\)

\(\Rightarrow\) cannot increase sample value of \(i\) any further

- we say that row \(i\) is maximized
Back to the example

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>a⁺</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>b⁺</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>-4</td>
<td>-1</td>
</tr>
</tbody>
</table>

• i = 2 (increase the value of c)
  • first only wi column of x
  (otherwise Tiv = 0 and no increase)
  • c is correlated only with x
  • could choose line of a⁺ or b⁺

Try pivot b⁺, x.

\[ x = b - y - 3 \]
\[ a⁺ = 1 + x - 4y = 1 + b - y - 3 - 4y = -2 + 6 - 5y \]

Does not have a satisfying sample point.

Translation on x axis to b takes us outside the solution set.

\[ x = a + y - 1 \]
\[ b = 3 + x + y = 3 + (a + y - 1) + y \]
\[ c = -4 - x = -4 - a - y + 1 \]

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<thead>
<tr>
<th></th>
<th>a⁺</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>b⁺</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>-3</td>
<td>-1</td>
</tr>
</tbody>
</table>

• Still cannot restrict
  c ≥ 0
  • Pivot again
Increase the value of \( c \).
Decrease either \( a^+ \) or \( y \).
But \( a^+ \) is at 0 and cannot be decreased.

Negative entries in \(+\)-restricted columns are not good.

Pivot in column of \( y \) and row of \( b^+ \).

\[
y = \frac{1}{2} \cdot (b - 2 - a) = \frac{1}{2} \cdot b - 1 - \frac{1}{2} \cdot a
\]

\[
x = -1 + a + y = -2 + \frac{1}{2} a + \frac{1}{2} b
\]

\[
c = -3 - a - \frac{1}{2} b + 1 + \frac{1}{2} a = -2 - \frac{1}{2} a - \frac{1}{2} b
\]

<table>
<thead>
<tr>
<th></th>
<th>( a^+ )</th>
<th>( b^+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>-2</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( y )</td>
<td>-1</td>
<td>-( \frac{1}{2} )</td>
</tr>
<tr>
<td>( c )</td>
<td>-2</td>
<td>-( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

\[ a = x - y + 1 \]
\[ b = 2 + a + 2 y \geq 0 \]
\[ c = -2 \]