Bounded Independence Fools Degree-2 Threshold Functions

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Abstract

Let x be a random vector coming from any k-wise independent distribution over $\{-1,1\}^n$. For an n-variate degree-2 polynomial p, we prove that $\mathbf{E}[\mathrm{sgn}(p(x))]$ is determined up to an additive ε for $k = \mathrm{poly}(1/\varepsilon)$. This answers an open question of Diakonikolas et al. (FOCS 2009). Using standard constructions of k-wise independent distributions, we obtain a broad class of explicit generators that ε-fool the class of degree-2 threshold functions with seed length $\log n \cdot \mathrm{poly}(1/\varepsilon)$.

Our approach is quite robust: it easily extends to yield that the intersection of any constant number of degree-2 threshold functions is ε-fooled by $\mathrm{poly}(1/\varepsilon)$-wise independence. Our results also hold if the entries of x are k-wise independent standard normals, implying for example that bounded independence derandomizes the Goemans-Williamson hyperplane rounding scheme.

To achieve our results, we introduce a technique we dub *multivariate FT-mollification*, a generalization of the univariate form introduced by Kane et al. (SODA 2010) in the context of streaming algorithms. Along the way we prove a generalized hypercontractive inequality for quadratic forms which takes the operator norm of the associated matrix into account. These techniques may be of independent interest.

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1 Introduction

This paper is concerned with the power of limited independence to fool low-degree polynomial threshold functions. A degree-d *polynomial threshold function* (henceforth PTF), is a boolean function $f : \{-1,1\}^n \to \{-1,1\}$ expressible as $f(x) = \mathrm{sgn}(p(x))$, where p is an n-variate degree-d polynomial with real coefficients, and sgn is -1 for negative arguments and 1 otherwise. PTFs have played an important role in computer science since the early perceptron work of Minsky and Papert [31], and have since been extensively investigated in circuit complexity and communication complexity [2, 6, 10, 11, 19, 22, 28, 34, 35, 37, 38], learning theory [26, 27, 39], and more.

A distribution $\mathcal{D}$ on $\{-1,1\}^n$ is said to ε-fool a function $f : \{-1,1\}^n \to \{-1,1\}$ if

$$|\mathbf{E}_{x\sim\mathcal{D}}[f(x)] - \mathbf{E}_{x\sim\mathcal{U}}[f(x)]| \le \varepsilon$$

where $\mathcal{U}$ is the uniform distribution on $\{-1,1\}^n$. A distribution $\mathcal{D}$ on $\{-1,1\}^n$ is k-wise independent if every restriction of $\mathcal{D}$ to k coordinates is uniform on $\{-1,1\}^k$. Despite their simplicity, k-wise independent distributions have been a surprisingly powerful and versatile derandomization tool, fooling complex functions such as AC^0 circuits [4, 36, 9] and half-spaces [14]. As a result, this class of distributions has played a fundamental role in many areas of theoretical computer science.

Our Results. The problem we study is the following: How large must $k = k(n, d, \varepsilon)$ be in order for *every* k-wise independent distribution on $\{-1,1\}^n$ to ε-fool the class of degree-d PTF's? The $d = 1$ case of this problem was recently considered in [14], where it was shown that $k(n, 1, \varepsilon) = \widetilde{\Theta}(1/\varepsilon^2)$, independent of n, with an alternative proof to much of the argument given in [25]. The main open problem in [14] was to identify $k = k(n, d, \varepsilon)$ for $d \ge 2$. In this work, we make progress on this question by proving the following:

Theorem 1.1. Any $\tilde{\Omega}(\varepsilon^{-9})$-wise independent distribution on $\{-1,1\}^n$ ε-fools all degree-2 PTFs.

Prior to this work, no nontrivial result was known for $d > 1$; it was not even known whether $o(n)$-wise independence suffices for constant ε. Using known constructions of k-wise independent distributions [1, 13], Theorem 1.1 gives a large class of pseudo-random generators (PRGs) for degree-2 PTFs with seed length $\log(n) \cdot \widetilde{O}(\varepsilon^{-9})$.

Our techniques are quite robust. Our approach yields for example that Theorem 1.1 holds not only over the hypercube, but also over the n-variate Gaussian distribution. This already implies that the Goemans-Williamson hyperplane rounding scheme [18] (henceforth "GW rounding") can be derandomized using $\mathrm{poly}(1/\varepsilon)$-wise independence[1]. Our technique also readily extends to show that the intersection of m halfspaces, or even m degree-2 threshold functions, is ε-fooled by $\mathrm{poly}(1/\varepsilon)$-wise independence for any constant m (over both the hypercube and the multivariate Gaussian). One consequence of this is that $O(1/\varepsilon^2)$-wise independence suffices for GW rounding.

Another consequence of Theorem 1.1 is that bounded independence suffices for the invariance principle of Mossell, O'Donnell, and Oleszkiewicz in the case degree-2 polynomials. Let $p(x)$ be an n-variate degree-2 multi-linear polynomial with "low influences". The invariance principle roughly says that the distribution of p is essentially invariant if x is drawn from the uniform distribution on $\{-1,1\}^n$ versus the standard n-dimensional Gaussian distribution $\mathcal{N}(0,1)^n$. Our result implies that the x's do not need to be fully independent for the invariance principle to apply, but that bounded independence suffices.

[1] We note that other derandomizations of GW rounding are known with better dependence on ε, though not solely using k-wise independence; see [29, 40].

Motivation and Related Work. The literature is rich with explicit generators for various natural classes of functions. Recently, there has been much interest in not only constructing PRGs for natural complexity classes, but also in doing so with as broad and natural a family of PRGs as possible. One example is the recent work of Bazzi [4] on fooling depth-2 circuits (simplified by Razborov [36]), and of Braverman [9] on fooling AC^0, with bounded independence[2].

Simultaneously and independently from our work, Meka and Zuckerman [30] constructed PRGs against degree-d PTFs with seed length $\log n \cdot 2^{O(d)} \cdot (1/\varepsilon)^{8d+3}$ [30]. That is, their seed length for $d = 2$ is similar to ours (though worse by a $\mathrm{poly}(1/\varepsilon)$ factor). However, their result is incomparable to ours since their pseudorandom generator is customized for PTFs, and not based on k-wise independence alone. We believe that the ideas in our proof may lead to generators with better seed-length[3], and that some of the techniques we introduce are of independent interest.

In other recent and independent works, [20, 23] give PRGs for intersections of m halfspaces (though not degree-2 threshold functions). The former has polynomial dependence on m and requires only bounded independence as well (and considers other functions of halfspaces beside intersections), while the latter has poly-logarithmic dependence on m under the Gaussian measure but is not solely via bounded independence. Our dependence on m is polynomial.

2 Notation

Let $p : \{-1,1\}^n \to \mathbb{R}$ be a polynomial and $p(x) = \sum_{S\subseteq[n]} \widehat{p}_S \chi_S$ be its Fourier-Walsh expansion, where $\chi_S(x) \stackrel{\text{def}}{=} \prod_{i\in S} x_i$. The *influence* of variable i on p is $\mathrm{Inf}_i(p) \stackrel{\text{def}}{=} \sum_{S\ni i} \widehat{p}_S^2$, and the *total influence* of p is $\mathrm{Inf}(p) = \sum_{i=1}^n \mathrm{Inf}_i(p)$. If $\mathrm{Inf}_i(p) \le \tau \cdot \mathrm{Inf}(p)$ for all i, we say that the polynomial p is τ-*regular*. If $f(x) = \mathrm{sgn}(p(x))$, where p is τ-regular, we say that f is a τ-*regular* PTF.

For $R \subseteq \mathbb{R}^d$ denote by $I_R : \mathbb{R}^d \to \{0,1\}$ its characteristic function. It will be convenient in some of the proofs to phrase our results in terms of ε-fooling $\mathbf{E}[I_{[0,\infty)}(p(x))]$ as opposed to $\mathbf{E}[\mathrm{sgn}(p(x))]$. It is straightforward that these are equivalent up to changing ε by a factor of 2.

We frequently use $A \approx_\varepsilon B$ to denote that $|A - B| = O(\varepsilon)$, and we let the function $d_2(x,R)$ denote the L_2 distance from some $x \in \mathbb{R}^d$ to a region $R \subseteq \mathbb{R}^d$.

3 Overview of our proof of Theorem 1.1

The program of our proof follows the outline of the proof in [14]. We first prove that bounded independence fools the class of *regular* degree-2 PTF's. We then reduce the general case to the regular case to show that bounded independence fools all degree-2 PTF's. The bulk of our proof is to establish the first step; this is the most challenging part of this work and where our main technical contribution lies. The second step is achieved by adapting the recent results of [15].

We now elaborate on the first step. Let $f : \{-1,1\}^n \to \{-1,1\}$ be a boolean function. To show that f is fooled by k-wise independence, it suffices – and is in fact necessary – to prove the existence of two degree-k "sandwiching" polynomials $q_u, q_l : \{-1,1\}^n \to \{-1,1\}$ that approximate f in a certain technical sense (see e.g. [4, 7]). Even though this is an n-dimensional approximation problem, it may be possible to exploit the additional structure of the function under consideration to reduce it to a low-dimensional problem. This is exactly what is done in both [14] and [25] for the case of regular halfspaces.

[2] Note that a PRG for AC^0 with qualitatively similar – in fact slightly better – seed length had being already given by Nisan [33].

[3] An easy probabilistic argument shows that there exists PRGs for degree-d PTFs with seed-length $O(d \log(n/\varepsilon))$.

We now briefly explain the approaches of [14] and [25]. Let $f(x) = \mathrm{sgn}(\langle w, x\rangle)$ be an ε^2-regular halfspace, i.e. $\|w\|_2 = 1$ and $\max_i |w_i| \le \varepsilon$. An insight used in [14] (and reused in [25]) is the following: the random variable $\langle w, x\rangle$ behaves approximately like a standard Gaussian, hence it can be treated as if it was one-dimensional. Thus, both [14] and [25] construct a (different in each case) univariate polynomial $P : \mathbb{R} \to \mathbb{R}$ that is a "good" approximation to the sign function under the normal distribution in $\mathbb{R}$ (in the case of [25], the main point of the alternative proof was to avoid explicitly reasoning about any such polynomials, but the existence of such a polynomial is still implicit in the proof). The desired n-variate sandwiching polynomials are then obtained (roughly) by setting $q_u(x) = P(\langle w, x\rangle)$ and $q_u(x) = -P(-\langle w, x\rangle)$. It turns out that this approach suffices for the case of halfspaces. In [14] the polynomial P is constructed using approximation theory arguments. In [25] it is obtained by taking a truncated Taylor expansion of a certain smooth approximation to the sign function, constructed via a method dubbed "Fourier Transform mollification" (henceforth FT-mollification). We elaborate in Section 3.1 below.

Let $f(x) = \mathrm{sgn}(p(x))$ be a regular degree-2 PTF. A first natural attempt to handle this case would be to use the univariate polynomial P described above – potentially allowing its degree to increase – and then take $q_u(x) = P(p(x))$, as before. Unfortunately, such an approach fails for both constructions outlined above. We elaborate on this issue in Section C.

3.1 FT-mollification FT-mollification is a general procedure to obtain a smooth function with bounded derivatives that approximates some bounded function f. The univariate version of the method in the context of derandomization was introduced in [25]. In this paper we generalize it to the multivariate setting and later use it to prove our main theorem.

For the univariate case, where $f : \mathbb{R} \to \mathbb{R}$, [25] defined $\tilde{f}^c(x) = (c \cdot \hat{b}(c \cdot t) * f(t))(x)$ for a parameter c, where $\hat{b}$ has unit integral and is the Fourier transform of a smooth function b of compact support (a so-called *bump function*). Here "$*$" denotes convolution. The idea of smoothing functions via convolution with a smooth approximation of the Dirac delta function is old, dating back to "Friedrichs mollifiers" [17] in 1944. Indeed, the only difference between Friedrichs mollification and FT-mollification is that in the former, one convolves f with the scaled bump function, and not its Fourier transform. The switch to the Fourier transform is made to have better control on the high-order derivatives of the resulting smooth function, which is crucial for our application.

In our context, the method can be illustrated as follows. Let $X = \sum_i a_i X_i$ for independent X_i. Suppose we would like to argue that $\mathbf{E}[f(X)] \approx_\varepsilon \mathbf{E}[f(Y)]$, where $Y = \sum_i a_i Y_i$ for k-wise independent Y_i's that are individually distributed as the X_i. Let $\tilde{f}^c$ be the FT-mollified version of f. If the parameter $c = c(\varepsilon)$ is appropriately selected, we can guarantee that $|f(x) - \tilde{f}^c(x)| < \varepsilon$ "almost everywhere", and furthermore have "good" upper bounds on the high-order derivatives of $\tilde{f}^c$. We could then hope to show the following chain of inequalities: $\mathbf{E}[f(X)] \approx_\varepsilon \mathbf{E}[\tilde{f}^c(X)] \approx_\varepsilon \mathbf{E}[\tilde{f}^c(Y)] \approx_\varepsilon \mathbf{E}[f(Y)]$. To justify the first inequality, note f and $\tilde{f}^c$ are close almost everywhere, and so it suffices to argue that X is sufficiently anti-concentrated in the small region where they are not close. The second inequality would use Taylor's theorem, bounding the error via upper bounds on moment expectations of X and the high-order derivatives of $\tilde{f}^c$. Showing the final inequality would be similar to the first, except that one needs to justify that even under k-wise independence the distribution of Y is sufficiently anti-concentrated. We note that the argument outlined above was used in [25] to provide an alternative proof that bounded independence fools regular halfspaces, and to optimally derandomize Indyk's moment estimation algorithm in data streams [24]. However, this univariate approach fails for degree-2 PTFs for technical reasons (see Section C).

We now describe our switch to multivariate FT-mollification. Let $f : \{-1,1\}^n \to \{-1,1\}$ be arbitrary and let $S \subset \mathbb{R}^n$ with $f^{-1}(1) \subseteq S \subseteq \mathbb{R}^n \backslash f^{-1}(-1)$. Then fooling $\mathbf{E}[f(x)]$ and fooling $\mathbf{E}[I_S(x)]$ are equivalent. A natural attempt to this end would be to generalize FT-mollification to n dimensions, then FT-mollify I_S and argue as above using the multivariate Taylor's theorem. Such an approach is perfectly valid, but as one might expect, there is a penalty for working over high dimensions. Both our quantitative bounds on the error introduced by FT-mollifying, and the error coming from the multivariate Taylor's theorem, increase with the dimension. Our approach is then to find a *low-dimensional representation* of such a region S which allows us to obtain the desired bounds. We elaborate below on how this can be accomplished in our setting.

3.2 Our Approach Let $f = \mathrm{sgn}(p)$ be a regular multilinear degree-2 PTF with $\|p\|_2 = 1$ (wlog). Let us assume for simplicity that p is a quadratic form; handling the additive linear form and constant is easy. The first conceptual step in our proof is this: we decompose p as $p_1 - p_2 + p_3$, where p_1, p_2 are positive semidefinite quadratic forms with no small non-zero eigenvalues and p_3 is indefinite with all eigenvalues small in magnitude. This decomposition, whose existence follows from elementary linear algebra, is particularly convenient for the following reason: for p_1, p_2, we are able to exploit their positive semidefiniteness to obtain better bounds from Taylor's theorem, and for p_3 we can establish moment bounds that are *strictly* stronger than the ones that follow from hypercontractivity for general quadratic forms (our Theorem 5.1, which may be of independent interest). The fact that we need p_1, p_2 to not only be positive semidefinite, but to also have no small eigenvalues, arises for technical reasons; specifically, quadratic forms with no small non-zero eigenvalues satisfy much better tail bounds, which plays a role in our analysis.

We now proceed to describe the second conceptual step of the proof, which involves multivariate FT-mollification. As suggested by the aforementioned, we would like to identify a region $R \subseteq \mathbb{R}^3$ such that $I_{[0,\infty)}(p(x))$ can be written as $I_R(F(x))$ for some $F : \{-1,1\}^n \to \mathbb{R}^3$ that depends on the p_i, then FT-mollify I_R. The region R is selected as follows: note we can write $p_3(x) = x^T A_{p_3} x$, where A_{p_3} is a real symmetric matrix with trace Υ. We consider the region $R = \{x : x_1^2 - x_2^2 + x_3 + \Upsilon \geq 0\} \subseteq \mathbb{R}^3$. Observe that $I_{[0,\infty)}(p(x)) = I_R(\sqrt{p_1(x)}, \sqrt{p_2(x)}, p_3(x) - \Upsilon)$. (Recall that p_1, p_2 are positive-semidefinite, hence the first two coordinates are always real.) We then prove via FT-mollification that $\mathbf{E}[I_R(\sqrt{p_1(x)}, \sqrt{p_2(x)}, p_3(x) - \Upsilon)]$ is preserved within ε by bounded independence.

The high-level argument is of similar flavor as the one outlined above for the case of halfspaces, but the details are more elaborate. The proof makes essential use of good tail bounds for p_1, p_2, a new moment bound for p_3, properties of FT-mollification, and a variety of other tools such as the Invariance Principle [32] and the anti-concentration bounds of [12].

Organization. Section 4 contains the results we will need on multivariate FT-mollification. In Section 5 we give our improved moment bound on quadratic forms. Section 6 contains the analysis of the regular case, and Section 7 concludes the proof of our main theorem. Section 8 summarizes our results on intersections.

4 Multivariate FT-mollification

Definition 4.1. In *hyperspherical coordinates* in $\mathbb{R}^d$, we represent a point $x = (x_1, \ldots, x_d)$ by $x_i = r\cos(\phi_i)\prod_{j=1}^{i-1}\sin(\phi_j)$ for $i < d$, and $x_d = r\prod_{j=1}^{d-1}\sin(\phi_j)$. Here $r = \|x\|_2$ and the ϕ_i satisfy $0 \leq \phi_i \leq \pi$ for $i < d-1$, and $0 \leq \phi_{d-1} < 2\pi$.

Fact 4.2. Let J be the Jacobian matrix corresponding to the change of variables from Cartesian to hyperspherical coordinates. Then

$$\det(J) = r^{d-1} \prod_{i=1}^{d-2} \sin^{d-1-i}(\phi_i).$$

We define the bump function $b : \mathbb{R}^d \to \mathbb{R}$ by

$$b(x) = \sqrt{C_d} \cdot \begin{cases} 1 - \|x\|_2^2 & \text{for } \|x\|_2 < 1 \\ 0 & \text{otherwise} \end{cases}.$$

The value C_d is chosen so that $\|b\|_2^2 = 1$. We note that b is not smooth (its mixed partials do not exist at the boundary of the unit sphere), but we will only ever need that $\frac{\partial}{\partial x_i} b \in L^2(\mathbb{R}^d)$ for all $i \in [d]$.

Henceforth, we make the setting

$$A_d = C_d \cdot \int_0^{2\pi} \int_{[0,\pi]^{d-2}} \left(\prod_{i=1}^{d-2} \sin^{d-1-i}(\phi_i) \right) d\phi_1 d\phi_2 \cdots d\phi_{d-1}.$$

We let $\hat{b} : \mathbb{R}^d \to \mathbb{R}$ denote the Fourier transform of b, i.e.

$$\hat{b}(t) = \frac{1}{(\sqrt{2\pi})^d} \int_{\mathbb{R}^d} b(x) e^{-i\langle x,t \rangle} dx.$$

Finally, $B : \mathbb{R}^d \to \mathbb{R}$ denotes the function $\hat{b}^2$, and we define $B_c : \mathbb{R}^d \to \mathbb{R}$ by

$$B_c(x_1, \ldots, x_d) = c^d \cdot B(cx_1, \ldots, cx_d).$$

Definition 4.3 (Multivariate FT-mollification). For $F : \mathbb{R}^d \to \mathbb{R}$ and given $c > 0$, we define the *FT-mollification* $\tilde{F}^c : \mathbb{R}^d \to \mathbb{R}$ by

$$\tilde{F}^c(x) = (B_c * F)(x) = \int_{\mathbb{R}^d} B_c(y) F(x - y) dy.$$

In this section we give several quantitative properties of FT-mollification. We start off with a few lemmas that will be useful later.

Lemma 4.4. For any $c > 0$,

$$\int_{\mathbb{R}^d} B_c(x) dx = 1.$$

Proof. Since $B = \hat{b}^2$, the stated integral when $c = 1$ is $\|\hat{b}\|_2^2$, which is $\|b\|_2^2 = 1$ by Plancherel's theorem. For general c, make the change of variables $u = (cx_1, \ldots, cx_d)$ then integrate over u. ■

Before presenting the next lemma, we familiarize the reader with some multi-index notation. A d-dimensional multi-index is a vector $\beta \in \mathbb{N}^d$ (here $\mathbb{N}$ is the nonnegative integers). For $\alpha, \beta \in \mathbb{N}^d$, we say $\alpha \le \beta$ if the inequality holds coordinate-wise, and for such α, β we define $|\beta| = \sum_i \beta_i$, $\binom{\beta}{\alpha} = \prod_{i=1}^d \binom{\beta_i}{\alpha_i}$, and $\beta! = \prod_{i=1}^d \beta_i!$. For $x \in \mathbb{R}^d$ we use x^β to denote $\prod_{i=1}^d x_i^{\beta_i}$, and for $f : \mathbb{R}^d \to \mathbb{R}$ we use $\partial^\beta f$ to denote $\frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} f$.

Lemma 4.5. For any $\beta \in \mathbb{N}^d$, $\|\partial^\beta B\|_1 \le 2^{|\beta|}$.

Proof. We have

$$\partial^\beta B = \sum_{\alpha \le \beta} \binom{\beta}{\alpha} \left(\partial^\alpha \hat{b}\right) \cdot \left(\partial^{\beta-\alpha} \hat{b}\right)$$

Thus,

$$\begin{aligned}
\left\|\partial^\beta B\right\|_1 &= \left\| \sum_{\alpha \le \beta} \binom{\beta}{\alpha} \left(\partial^\alpha \hat{b}\right) \cdot \left(\partial^{\beta-\alpha} \hat{b}\right) \right\|_1 \\
&\le \sum_{\alpha \le \beta} \binom{\beta}{\alpha} \left\|\partial^\alpha \hat{b}\right\|_2 \cdot \left\|\partial^{\beta-\alpha} \hat{b}\right\|_2 && (4.1) \\
&= \sum_{\alpha \le \beta} \binom{\beta}{\alpha} \left\|x^\alpha \cdot b\right\|_2 \cdot \left\|x^{\beta-\alpha} \cdot b\right\|_2 && (4.2) \\
&\le \sum_{\alpha \le \beta} \binom{\beta}{\alpha} && (4.3) \\
&= 2^{|\beta|} && (4.4)
\end{aligned}$$

Eq. (4.1) follows by Cauchy-Schwarz. Eq. (4.2) follows from Plancherel's theorem, since the Fourier transform of $\partial^\alpha \hat{b}$ is $x^\alpha \cdot b$, up to factors of i. Eq. (4.3) follows since $\|x^\alpha \cdot b\|_2 \le \|b\|_2 = 1$. Eq. (4.4) is seen combinatorially. Suppose we have $2d$ buckets A_i^j for $(i,j) \in [d] \times [2]$. We also have $|\beta|$ balls, with each having one of d types with β_i balls of type i. Then the number of ways to place balls into buckets such that balls of type i only go into some A_i^j is $2^{|\beta|}$ (each ball has 2 choices). However, it is also $\sum_{\alpha \le \beta} \binom{\beta}{\alpha}$, since for every placement of balls we must place some number α_i balls of type i in A_i^1 and $\beta_i - \alpha_i$ balls in A_i^2. ■

Lemma 4.6. Let $z > 0$ be arbitrary. Then

$$\int_{\|x\|_2 > dz} B(x)dx = O(1/z^2).$$

Proof. Consider the integral

$$S = \int_{\mathbb{R}^d} \|x\|_2^2 \cdot B(x)dx = \sum_{i=1}^d \left(\int_{\mathbb{R}^d} x_i^2 \cdot B(x)dx \right).$$

Recalling that $B = \hat{b}^2$, the Fourier transform of B is $(2\pi)^{-d/2}(b * b)$. The above integral is $(2\pi)^{d/2}$ times the Fourier transform of $x_i^2 \cdot B$, evaluated at 0. Since multiplying a function by $i \cdot x_j$ corresponds to partial differentiation by x_j in the Fourier domain,

$$S = \sum_{i=1}^d \left(\frac{\partial^2}{\partial x_i^2}(b * b) \right)(0) = \sum_{i=1}^d \left(\left(\frac{\partial}{\partial x_i} b\right) * \left(\frac{\partial}{\partial x_i} b\right) \right)(0) = \sum_{i=1}^d \left\| \frac{\partial}{\partial x_i} b \right\|_2^2$$

with the last equality using that $\frac{\partial}{\partial x_i} b$ is odd.

We have, for x in the unit ball,

$$\left(\frac{\partial}{\partial x_i} b\right)(x) = -2x_i$$

so that, after switching to hyperspherical coordinates,

$$\sum_{i=1}^{d} \left\| \frac{\partial}{\partial x_i} b \right\|_2^2 = A_d \cdot \int_0^1 4r^{d+1} dr. \tag{4.5}$$

Claim 4.7.

$$\sum_{i=1}^{d} \left\| \frac{\partial}{\partial x_i} b \right\|_2^2 = O(d^2)$$

Proof. By definition of b,

$$\begin{aligned} \|b\|_2^2 &= A_d \cdot \int_0^1 r^{d-1} + r^{d+3} - 2r^{d+1} dr \\ &= A_d \cdot \frac{8}{d(d+2)(d+4)} \\ &= A_d \cdot \Omega(1/d^3). \end{aligned}$$

We also have by Eq. (4.5) that

$$\sum_{i=1}^{d} \left\| \frac{\partial}{\partial x_i} b \right\|_2^2 = A_d \cdot \frac{4}{d+2} = A_d \cdot O(1/d).$$

The claim follows since $\|b\|_2^2 = 1$.

■

We now finish the proof of the lemma. Since B has unit integral on $\mathbb{R}^d$ (Lemma 4.4) and is nonnegative everywhere, we can view B as the density function of a probability distribution on $\mathbb{R}^d$. Then S can be viewed as $\mathbf{E}_{x\sim B}[\|x\|_2^2]$. Then by Markov's inequality, for $x \sim B$,

$$\mathbf{Pr}\left[\|x\|_2^2 \geq z^2 \cdot \mathbf{E}[\|x\|_2^2]\right] \leq 1/z^2,$$

which is equivalent to

$$\mathbf{Pr}\left[\|x\|_2 \geq z \cdot \sqrt{\mathbf{E}[\|x\|_2^2]}\right] \leq 1/z^2.$$

We conclude by observing that the above probability is simply

$$\int_{\|x\|_2 \geq z \cdot \sqrt{\mathbf{E}[\|x\|_2^2]}} B(x) dx,$$

from which the lemma follows since $\mathbf{E}[\|x\|_2^2] = O(d^2)$ by Claim 4.7. ■

We now state the main theorem of this section, which says that if F is bounded, then $\tilde{F}^c$ is smooth with strong bounds on its mixed partial derivatives, and is close to F on points where F satisfies some continuity property.

Theorem 4.8. Let $F : \mathbb{R}^d \to \mathbb{R}$ be bounded and $c > 0$ be arbitrary. Then,

i. $\|\partial^\beta \tilde{F}^c\|_\infty \leq \|F\|_\infty \cdot (2c)^{|\beta|}$ for all $\beta \in \mathbb{N}^d$.

ii. Fix some $x \in \mathbb{R}^d$. Then if $|F(x) - F(y)| \leq \varepsilon$ whenever $\|x - y\|_2 \leq \delta$ for some $\varepsilon, \delta \geq 0$, then $|\tilde{F}^c(x) - F(x)| \leq \varepsilon + \|F\|_\infty \cdot O(d^2/(c^2\delta^2))$.

Proof. We first prove (i).

$$\begin{aligned}
\left|\left(\partial^\beta \tilde{F}^c\right)(x)\right| &= \left|\left(\partial^\beta (B_c * F)\right)(x)\right| \\
&= \left|\left(\left(\partial^\beta B_c\right) * F\right)(x)\right| \\
&= \left|\int_{\mathbb{R}^d} \left(\partial^\beta B_c\right)(y) F(x-y) dy\right| \\
&\leq \|F\|_\infty \cdot \left\|\partial^\beta B_c\right\|_1 \\
&= \|F\|_\infty \cdot c^{|\beta|} \cdot \left\|\partial^\beta B\right\|_1 \qquad (4.6) \\
&\leq \|F\|_\infty \cdot (2c)^{|\beta|}
\end{aligned}$$

with the last inequality holding by Lemma 4.5.

We now prove (ii).

$$\begin{aligned}
\tilde{F}^c(x) &= (B_c * F)(x) \\
&= \int_{\mathbb{R}^d} B_c(x-y)F(y)dy \\
&= F(x) + \int_{\mathbb{R}^d} (F(y) - F(x))B_c(x-y)dy \qquad (4.7) \\
&= F(x) + \int_{\|x-y\|_2 < \delta} (F(y) - F(x))B_c(x-y) + \int_{\|x-y\|_2 \geq \delta} (F(y) - F(x))B_c(x-y) \\
&= F(x) \pm \varepsilon \cdot \int_{\|x-y\|_2 < \delta} |B_c(x-y)| + \int_{\|x-y\|_2 \geq \delta} (F(y) - F(x))B_c(x-y) \\
&= F(x) \pm \varepsilon \cdot \int_{\mathbb{R}^d} B_c(x-y) + \int_{\|x-y\|_2 \geq \delta} (F(y) - F(x))B_c(x-y) \\
&= F(x) \pm \varepsilon \pm \|F\|_\infty \cdot \int_{\|x-y\|_2 \geq \delta} B_c(x-y)dy \\
&= F(x) \pm \varepsilon \pm \|F\|_\infty \cdot \int_{\|u\|_2 \geq c\delta} B(u)du \\
&= F(x) \pm \varepsilon \pm \|F\|_\infty \cdot O(d^2/(c^2\delta^2))
\end{aligned}$$

where Eq. (4.7) uses Lemma 4.4. ■

Remark 4.9. It is possible to obtain sharper bounds on $\|\partial^\beta \tilde{F}^c\|_\infty$. In particular, note in the proof of Theorem 4.8 that $\|\partial^\beta \tilde{F}^c\|_\infty \leq \|F\|_\infty \cdot c^{|\beta|} \cdot \|\partial^\beta B\|_1$. An improved bound on $\|\partial^\beta B\|_1$ versus that of Lemma 4.5 turns out to be possible. This improvement is useful when FT-mollifying over high dimension, but in the proof of our main result (Theorem 1.1) we are never concerned with $d > 4$. We thus above presented a simpler proof for clarity of exposition, and we defer the details of the improvement to Section G.1.

The following theorem is immediate from Theorem 4.8, and gives guarantees when FT-mollifying the indicator function of some region. In Theorem 4.10, and some later proofs which invoke the theorem, we use the following notation. For $R \subset \mathbb{R}^d$, we let ∂R denote the boundary of R (specifically in this context, ∂R is the set of points $x \in \mathbb{R}^d$ such that for every $\varepsilon > 0$, the ball about x of radius ε intersects both R and $\mathbb{R}^d \backslash R$).

Theorem 4.10. For any region $R \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

$$|I_R(x) - \tilde{I}_R^c(x)| \le \min\left\{1, O\left(\left(\frac{d}{c \cdot d_2(x, \partial R)}\right)^2\right)\right\}.$$

Proof. We have $|I_R(x) - \tilde{I}_R^c(x)| \le 1$ always. This follows since $\tilde{I}_R^c$ is nonnegative (it is the convolution of nonnegative functions), and is never larger than $\|I_R\|_\infty = 1$. The other bound is obtained, for $x \notin \partial R$, by applying Theorem 4.8 to $F = I_R$ with $\varepsilon = 0$, $\delta = d_2(x, \partial R)$. ■

5 A spectral moment bound for quadratic forms

For a quadratic form $p(x) = \sum_{i \le j} a_{i,j} x_i x_j$, we can associate a real symmetric matrix A_p which has the $a_{i,i}$ on the diagonals and $a_{\min\{i,j\},\max\{i,j\}}/2$ on the offdiagonals, so that $p(x) = x^T A_p x$. We now show a moment bound for quadratic forms which takes into account the maximum eigenvalue of A_p. Our proof is partly inspired by a proof of Whittle [42], who showed the hypercontractive inequality for degree-2 polynomials when comparing q-norms to 2-norms (see Theorem B.1).

Recall the *Frobenius norm* of $A \in \mathbb{R}^{n \times n}$ is $\|A\|_2 = \sqrt{\sum_{i,j=1}^{n,n} A_{i,j}^2} = \sqrt{\sum_i \lambda_i^2} = \sqrt{\mathrm{tr}(A^2)}$, where tr denotes trace and A has eigenvalues $\lambda_1, \ldots, \lambda_n$. Also, let $\|A\|_\infty$ be the largest magnitude of an eigenvalue of A. We can now state and prove the main theorem of this section, which plays a crucial role in our analysis of the regular case of our main theorem (Theorem 1.1).

Theorem 5.1. Let $A \in \mathbb{R}^{n \times n}$ be symmetric and $x \in \{-1, 1\}^n$ be random. Then for all $k \ge 2$,

$$\mathbf{E}[|(x^T A x) - \mathrm{tr}(A)|^k] \le C^k \cdot \max\{\sqrt{k}\|A\|_2, k\|A\|_\infty\}^k$$

where C is an absolute constant.

Note if $\sum_{i \le j} a_{i,j}^2 \le 1$ then $\|A_p\|_\infty \le 1$, in which case our bound recovers a similar moment bound as the one obtained via hypercontractivity. Thus, in the special case of bounding kth moments of degree-2 polynomials against their 2nd moment, our bound can be viewed as a generalization of the hypercontractive inequality (and of Whittle's inequality).

We first give two lemmas. The first is implied by Khintchine's inequality [21], and the second is a discrete analog of one of Whittle's lemmas.

Lemma 5.2. For $a \in \mathbb{R}^n$, x as above, and $k \ge 2$ an even integer, $\mathbf{E}[(a^T x)^k] \le \|a\|_2^k \cdot k^{k/2}$.

Lemma 5.3. If X, Y are independent with $\mathbf{E}[Y] = 0$ and if $k \ge 2$, then $\mathbf{E}[|X|^k] \le \mathbf{E}[|X - Y|^k]$.

Proof. Consider the function $f(y) = |X - y|^k$. Since $f^{(2)}$, the second derivative of f, is nonnegative on $\mathbb{R}$, the claim follows by Taylor's theorem since $|X - Y|^k \ge |X|^k - kY(\mathrm{sgn}(X) \cdot X)^{k-1}$. ■

We are now prepared to prove our Theorem 5.1.

Proof (of Theorem 5.1). Without loss of generality we can assume $\mathrm{tr}(A) = 0$. This is because if one considers $A' = A - (\mathrm{tr}(A)/n) \cdot I$, then $x^T A x - \mathrm{tr}(A) = x^T A' x$, and we have $\|A'\|_2 \le \|A\|_2$ and $\|A'\|_\infty \le 2\|A\|_\infty$. We now start by proving our theorem for k a power of 2 by induction on k. For

$k = 2$, $\mathbf{E}[(x^T Ax)^2] = 4\sum_{i<j} A_{i,j}^2$ and $\|A\|_2^2 = \sum_i A_{i,i}^2 + 2\sum_{i<j} A_{i,j}^2$. Thus $\mathbf{E}[(x^T Ax)^2] \le 2\|A\|_2^2$. Next we assume the statement of our Theorem for $k/2$ and attempt to prove it for k.

We note that by Lemma 5.3,

$$\mathbf{E}[|x^T Ax|^k] \le \mathbf{E}[|x^T Ax - y^T Ay|^k] = \mathbf{E}[|(x+y)^T A(x-y)|^k],$$

where $y \in \{-1,1\}^n$ is random and independent of x. Notice that if we swap x_i with y_i then $x+y$ remains constant as does $|x_j - y_j|$ and that $x_i - y_i$ is replaced by its negation. Consider averaging over all such swaps. Let $\xi_i = ((x+y)^T A)_i$ and $\eta_i = x_i - y_i$. Let z_i be 1 if we did not swap and -1 if we did. Then $(x+y)^T A(x-y) = \sum_i \xi_i \eta_i z_i$. Averaging over all swaps,

$$\mathbf{E}_z[|(x+y)^T A(x-y)|^k] \le \left(\sum_i \xi_i^2 \eta_i^2\right)^{k/2} \cdot k^{k/2} \le 2^k k^{k/2} \cdot \left(\sum_i \xi_i^2\right)^{k/2}.$$

The first inequality is by Lemma 5.2, and the second uses that $|\eta_i| \le 2$. Note that

$$\sum_i \xi_i^2 = \|A(x+y)\|_2^2 \le 2\|Ax\|_2^2 + 2\|Ay\|_2^2,$$

and hence

$$\mathbf{E}[|x^T Ax|^k] \le 2^k \sqrt{k}^k \mathbf{E}[(2\|Ax\|_2^2 + 2\|Ay\|_2^2)^{k/2}] \le 4^k \sqrt{k}^k \mathbf{E}[(\|Ax\|_2^2)^{k/2}],$$

with the final inequality using Minkowski's inequality (namely that $|\mathbf{E}[|X+Y|^p]|^{1/p} \le |\mathbf{E}[|X|^p]|^{1/p} + |\mathbf{E}[|Y|^p]|^{1/p}$ for any random variables X, Y and any $1 \le p < \infty$).

Next note $\|Ax\|_2^2 = \langle Ax, Ax\rangle = x^T A^2 x$. Let $B = A^2 - \frac{\mathrm{tr}(A^2)}{n} I$. Then $\mathrm{tr}(B) = 0$. Also, $\|B\|_2 \le \|A\|_2 \|A\|_\infty$ and $\|B\|_\infty \le \|A\|_\infty^2$. The former holds since

$$\|B\|_2^2 = \left(\sum_i \lambda_i^4\right) - \left(\sum_i \lambda_i^2\right)^2 \Big/ n \le \sum_i \lambda_i^4 \le \|A\|_2^2 \|A\|_\infty^2.$$

The latter holds since the eigenvalues of B are $\lambda_i^2 - (\sum_{j=1}^n \lambda_j^2)/n$ for each $i \in [n]$. The largest eigenvalue of B is thus at most that of A^2, and since $\lambda_i^2 \ge 0$, the smallest eigenvalue of B cannot be smaller than $-\|A\|_\infty^2$.

We then have

$$\mathbf{E}[(\|Ax\|_2^2)^{k/2}] = \mathbf{E}\left[\left|\|A\|_2^2 + x^T Bx\right|^{k/2}\right] \le 2^k \max\{\|A\|_2^k, \mathbf{E}[|x^T Bx|^{k/2}]\}.$$

Hence employing the inductive hypothesis on B we have that

$$\begin{aligned}
\mathbf{E}[|x^T Ax|^k] &\le 8^k \max\{\sqrt{k}\|A\|_2, C^{k/2} k^{3/4} \|B\|_2, C^{k/2} k \sqrt{\|B\|_\infty}\}^k \\
&\le 8^k C^{k/2} \max\{\sqrt{k}\|A\|_2, k^{3/4} \sqrt{\|A\|_2 \|A\|_\infty}, k\|A\|_\infty\}^k \\
&= 8^k C^{k/2} \max\{\sqrt{k}\|A\|_2, k\|A\|_\infty\}^k,
\end{aligned}$$

with the final equality holding since the middle term above is the geometric mean of the other two, and thus is dominated by at least one of them. This proves our hypothesis as long as $C \ge 64$.

To prove our statement for general k, set $k' = 2^{\lceil \log_2 k \rceil}$. Then by the power mean inequality and our results for k' a power of 2, $\mathbf{E}[|x^T Ax|^k] \le (\mathbf{E}[|x^T Ax|^{k'}])^{k/k'} \le 128^k \max\{\sqrt{k}\|A\|_2, k\|A\|_\infty\}^k$. ■

6 Fooling regular degree-2 threshold functions

The main theorem of this section is the following.

Theorem 6.1. Let $0 < \varepsilon < 1$ be given. Let $X_1, \ldots, X_n$ be independent Bernoulli and $Y_1, \ldots, Y_n$ be $2k$-wise independent Bernoulli for k a sufficiently large multiple of $1/\varepsilon^8$. If p is multilinear and of degree 2 with $\sum_{|S|>0} \widehat{p}_S^2 = 1$, and $\mathrm{Inf}_i(p) \le \tau$ for all i, then

$$\mathbf{E}[\mathrm{sgn}(p(X))] - \mathbf{E}[\mathrm{sgn}(p(Y))] = O(\varepsilon + \tau^{1/9}).$$

Throughout this section, p always refers to the polynomial of Theorem 6.1, and τ refers to the maximum influence of any variable in p. Observe p (over the hypercube) can be written as $q+p_4+C$, where q is a multilinear quadratic form, p_4 is a linear form, and C is a constant. Furthermore, $\|A_q\|_2 \le 1/2$ and $\sum_S \widehat{p_4}_S^2 \le 1$. Using the spectral theorem for real symmetric matrices, we write $p = p_1 - p_2 + p_3 + p_4 + C$ where p_1, p_2, p_3 are quadratic forms satisfying $\lambda_{\min}(A_{p_1}), \lambda_{\min}(A_{p_2}) \ge \delta$, $\|A_{p_3}\|_\infty < \delta$, and $\|A_{p_i}\|_2 \le 1/2$ for $1 \le i \le 3$, and also with p_1, p_2 positive semidefinite (see Lemma B.7 for details on how this is accomplished). Here $\lambda_{\min}(A)$ denotes the smallest magnitude of a non-zero eigenvalue of A. Throughout this section we let $p_1, \ldots, p_4$, C, δ be as discussed here. We use Υ to denote $\mathrm{tr}(A_{p_3})$. The value δ will be set later in the proof of Theorem 6.1.

Throughout this section it will be notationally convenient to define the map $M_p : \mathbb{R}^n \to \mathbb{R}^4$ by $M_p(x) = (\sqrt{p_1(x)}, \sqrt{p_2(x)}, p_3(x) - \Upsilon, p_4(x))$. Note the the first two coordinates of $M_p(x)$ are indeed always real since p_1, p_2 are positive semidefinite.

Before giving the proof of Theorem 6.1, we first prove Lemma 6.3, which states that for $F : \mathbb{R}^4 \to \mathbb{R}$, $F(M_p(x))$ is fooled by bounded independence as long as F is even in x_1, x_2 and certain technical conditions are satisfied. The proof of Lemma 6.3 invokes the following lemma, which follows from lemmas in the Appendix (specifically, by combining Lemma A.6 and Lemma B.5).

Lemma 6.2. For a quadratic form f and random $x \in \{-1,1\}^n$,

$$\mathbf{E}[|f(x)|^k] \le 2^{O(k)} \cdot (\|A_f\|_2 k^k + (\|A_f\|_2^2/\lambda_{\min}(A_f))^k).$$

Lemma 6.3. Let $\varepsilon > 0$ be arbitrary. Let $F : \mathbb{R}^4 \to \mathbb{R}$ be even in each of its first two arguments such that $\|\partial^\beta \tilde{F}^c\|_\infty = O(\alpha^{|\beta|})$ for all multi-indices $\beta \in \mathbb{N}^4$ and some $\alpha > 1$. Suppose $1/\delta \ge B\alpha$ for a sufficiently large constant B. Let $X_1, \ldots, X_n$ be independent Bernoulli, and $Y_1, \ldots, Y_n$ be k'-independent Bernoulli for $k' = 2k$ with $k \ge \max\{\log(1/\varepsilon), B\alpha/\sqrt{\delta}, B\alpha^2\}$ an even integer. Write $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$. Then $|\mathbf{E}[F(M_p(X))] - \mathbf{E}[F(M_p(Y))]| < \varepsilon$.

Proof. We Taylor-expand F to obtain a polynomial P_{k-1} containing all monomials up to degree $k-1$. Since $F(x)$ is even in x_1, x_2, we can assume P_{k-1} is a polynomial in x_1^2, x_2^2, x_3, x_4. Let $x \in \mathbb{R}^4$ be arbitrary. We apply Taylor's theorem to bound $R(x) = |F(x) - P_{k-1}(x)|$. Define $x_* = \max_i\{|x_i|\}$. Then

$$\begin{aligned} R(x) &\le \alpha^k \cdot \sum_{|\beta|=k} \frac{|x_1|^{\beta_1} \cdot |x_2|^{\beta_2} \cdot |x_3|^{\beta_3} \cdot |x_4|^{\beta_4}}{\beta_1! \cdot \beta_2! \cdot \beta_3! \cdot \beta_4!} \\ &\le \alpha^k x_*^k \cdot \sum_{|\beta|=k} \frac{1}{\beta_1! \cdot \beta_2! \cdot \beta_3! \cdot \beta_4!} \\ &= \alpha^k x_*^k \cdot \frac{1}{k!} \cdot \sum_{|\beta|=k} \binom{k}{\beta_1, \ldots, \beta_4} \end{aligned}$$

$$\leq \quad \alpha^k 4^k \cdot \frac{x_1^k + x_2^k + x_3^k + x_4^k}{k!}, \tag{6.1}$$

with the absolute values unnecessary in the last inequality since k is even. We now observe

$$|\mathbf{E}[F(M_p(X))] - \mathbf{E}[F(M_p(Y))]|$$
$$\leq \alpha^k 2^{O(k)} \cdot \frac{\mathbf{E}[(p_1(X))^{k/2}] + \mathbf{E}[(p_2(X))^{k/2}] + \mathbf{E}[(p_3(X) - \Upsilon)^k] + \mathbf{E}[(p_4(X))^k]}{k^k}$$

since (a) every term in $P_{k-1}(M_p(X))$ is a monomial of degree at most $2k-2$ in the X_i, by evenness of P_{k-1} in x_1, x_2, and is thus determined by $2k$-independence, (b) $\sqrt{p_1(X)}, \sqrt{p_2(X)}$ are real by positive semidefiniteness of p_1, p_2 (note that we are only given that the high order partial derivatives are bounded by $O(\alpha^k)$ on the reals; we have no guarantees for complex arguments), and (c) the moment expectations above are equal for X and Y since they are determined by $2k$-independence.

We now bound the error term above. We have

$$\mathbf{E}[(p_1(X))^{k/2}] = 2^{O(k)}(k^{k/2} + \delta^{-k/2})$$

by Lemma 6.2, with the same bound holding for $\mathbf{E}[(p_2(X))^{k/2}]$. We also have

$$\mathbf{E}[(p_3(X) - \Upsilon)^k] \leq 2^{O(k)} \cdot \max\{\sqrt{k}, (\delta k)\}^k$$

by Theorem 5.1. We finally have

$$\mathbf{E}[(p_4(X))^k] \leq k^{k/2}$$

by Lemma 5.2. Thus in total,

$$|\mathbf{E}[F(M_p(X))] - \mathbf{E}[F(M_p(Y))]| \leq 2^{O(k)} \cdot ((\alpha/\sqrt{k})^k + (\alpha/(k\sqrt{\delta}))^k + (\alpha\delta)^k),$$

which is at most ε for sufficiently large B by our lower bounds on k and $1/\delta$. ■

In proving Theorem 6.1, we will need a lemma which states that p is anticoncentrated even when evaluated on Bernoulli random variables which are k-wise independent. To show this, we make use of the following lemma, which follows from the Invariance Principle, the hypercontractive inequality, and the anticoncentration bound of [12]. The proof is in Section D.

Lemma 6.4. Let $\eta, \eta' \geq 0, t \in \mathbb{R}$ be given, and let $X_1, \ldots, X_n$ be independent Bernoulli. Then

$$\mathbf{Pr}[|p(X) - t| \leq \eta \cdot (\sqrt{p_1(X)} + \sqrt{p_2(X)} + 1) + \eta'] = O(\sqrt{\eta'} + (\eta^2/\delta)^{1/4} + \tau^{1/9} + \exp(-\Omega(1/\delta))).$$

We now prove our anticoncentration lemma in the case of limited independence.

Lemma 6.5. Let ε' be given. Suppose $k \geq D/(\varepsilon')^4$ for a sufficiently large constant $D > 0$. Let $Y_1, \ldots, Y_n$ be k-wise independent Bernoulli, and let $t \in \mathbb{R}$ be arbitrary. Then

$$\mathbf{Pr}[|p(Y) - t| < \varepsilon'] \leq O(\sqrt{\varepsilon'} + \tau^{1/9}).$$

Proof. Define the region $T_{t,\varepsilon'} = \{(x_1, x_2, x_3, x_4) : |x_1^2 - x_2^2 + x_3 + x_4 + C + \Upsilon - t| < \varepsilon'\}$, and also the region $S_{\rho,t,\varepsilon'} = \{x : d_2(x, T_{t,\varepsilon'}) \leq \rho\}$ for $\rho \geq 0$. Consider the FT-mollification $\tilde{I}^c_{S_{\rho,t,\varepsilon'}}$ of $I_{S_{\rho,t,\varepsilon'}}$ for $c = A/\rho$, with A a large constant to be determined later. We note a few properties of $\tilde{I}^c_{S_{\rho,t,\varepsilon'}}$:

i. $\|\partial^\beta \tilde{I}^c_{S_{\rho,t,\varepsilon'}}\|_\infty \leq (2c)^{|\beta|}$

ii. $\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(x) \geq \frac{1}{2} \cdot I_{T_{t,\varepsilon'}}(x)$

iii. $\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(x) = \max\left\{1, O\left((c \cdot d_2(x, T_{t,\varepsilon'}))^{-2}\right)\right\}$ for any x with $d_2(x, T_{t,\varepsilon'}) \geq 2\rho$

Item (i) is straightforward from Theorem 4.8. For item (ii), note that if $x \in T_{t,\varepsilon'}$, then $d_2(x, \partial S_{\rho,t,\varepsilon'}) \geq \rho$, implying

$$|\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(x) - 1| = O\left(\frac{1}{c^2\rho^2}\right),$$

which is at most $1/2$ for A a sufficiently large constant. Furthermore, $\tilde{I}^c_{S_{\rho,t,\varepsilon'}}$ is nonnegative. Finally, for (iii), by Theorem 4.10 we have

$$\begin{aligned}\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(x) &= \max\left\{1, O\left((c \cdot d_2(x, \partial S_{\rho,t,\varepsilon'}))^{-2}\right)\right\}\\ &\leq \max\left\{1, O\left((c \cdot d_2(x, S_{\rho,t,\varepsilon'}))^{-2}\right)\right\}\\ &\leq \max\left\{1, O\left((c \cdot (d_2(x, T_{t,\varepsilon'}) - \rho))^{-2}\right)\right\}\\ &\leq \max\left\{1, O\left((c \cdot d_2(x, T_{t,\varepsilon'}))^{-2}\right)\right\}\end{aligned}$$

with the last inequality using that $d_2(x, T_{t,\varepsilon'}) \geq 2\rho$.

Noting $\mathbf{Pr}[|p(Z) - t| < \varepsilon'] = \mathbf{E}[I_{T_{t,\varepsilon'}}(M_p(Z))]$ for any random variable $Z = (Z_1, \ldots, Z_n)$, item (ii) tells us that

$$\mathbf{Pr}[|p(Z) - t| \leq \varepsilon'] \leq 2 \cdot \mathbf{E}[\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(M_p(Z))]. \tag{6.2}$$

We now proceed in two steps. We first show $\mathbf{E}[\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(M_p(X))] = O(\sqrt{\varepsilon'} + \tau^{1/9})$ by applications of Lemma 6.4. We then show $\mathbf{E}[\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(M_p(Y))] = O(\sqrt{\varepsilon'} + \tau^{1/9})$ by applying Lemma 6.3, at which point we will have proven our lemma via Eq. (6.2) with $Z = Y$.

$\mathbf{E}[\tilde{\mathbf{I}}^{\mathbf{c}}_{\mathbf{S}_{\rho,\mathbf{t},\varepsilon'}}(\mathbf{M_p}(\mathbf{X}))] = \mathbf{O}(\sqrt{\varepsilon'} + \tau^{\mathbf{1/9}})$: We first observe that for $x \notin T_{t,\varepsilon'}$,

$$d_2(x, T_{t,\varepsilon'}) \geq \frac{1}{2} \cdot \min\left\{\frac{|x_1^2 - x_2^2 + x_3 + x_4 + C + \Upsilon - t| - \varepsilon'}{2(|x_1| + |x_2| + 1)}, \sqrt{|x_1^2 - x_2^2 + x_3 + x_4 + C + \Upsilon - t| - \varepsilon'}\right\}. \tag{6.3}$$

This is because by adding a vector v to x, we can change each individual coordinate of x by at most $\|v\|_2$, and can thus change the value of $|x_1^2 - x_2^2 + x_3 + x_4 + C + \Upsilon - t| - \varepsilon'$ by at most $2\|v\|_2 \cdot (|x_1| + |x_2| + 1) + \|v\|_2^2$.

Now let $X \in \{-1, 1\}^n$ be uniformly random. We thus have that, for any particular $w > 0$,

$$\begin{aligned}\mathbf{Pr}[0 < d_2(M_p(X), T_{t,\varepsilon'}) \leq w] &\leq \mathbf{Pr}\left[\min\left\{\frac{|p(X) - t| - \varepsilon'}{2(\sqrt{p_1(X)} + \sqrt{p_2(X)} + 1)}, \sqrt{|p(X) - t| - \varepsilon'}\right\} \leq 2w\right]\\ &\leq \mathbf{Pr}[|p(X) - t| \leq 4w \cdot (\sqrt{p_1}(X) + \sqrt{p_2(X)} + 1) + \varepsilon']\\ &\quad + \mathbf{Pr}[|p(X) - t| \leq 4w^2 + \varepsilon']\\ &= O(\sqrt{\varepsilon'} + w + \sqrt{w} + (w^2/\delta)^{1/4} + \tau^{1/9} + \exp(-\Omega(1/\delta)))\end{aligned}$$

with the last inequality holding by Lemma 6.4.

Now, by item (iii),

$$\begin{aligned}
&\mathbf{E}[\tilde{I}^c{}_{S_{\rho,t,\varepsilon'}}(M_p(X))] \\
&\quad \le \mathbf{Pr}[d_2(M_p(X),T_{t,\varepsilon'}) \le 2\rho] + O\left(\sum_{s=1}^{\infty} 2^{-2s}\cdot \mathbf{Pr}[2^s\rho < d_2(M_p(X),T_{t,\varepsilon'}) \le 2^{s+1}\rho]\right) \\
&\quad \le O(\sqrt{\varepsilon'}+\sqrt{\rho}+(\rho^2/\delta)^{1/4}+\tau^{1/9}+\exp(-\Omega(1/\delta)) \\
&\qquad + O\left(\sum_{s=1}^{\infty} 2^{-2s}\cdot(\sqrt{\varepsilon'}+2^{s+1}\rho+\sqrt{2^{s+1}\rho}+(2^{2s+2}\rho^2/\delta)^{1/4}+\tau^{1/9}+\exp(-\Omega(1/\delta)))\right) \\
&\quad = O(\sqrt{\varepsilon'}+\sqrt{\rho}+(\rho^2/\delta)^{1/4}+\tau^{1/9}+\exp(-\Omega(1/\delta)) \qquad (6.4)
\end{aligned}$$

We now make the settings

$$\rho = (\varepsilon')^2, \qquad \frac{1}{\delta} = 2Bc = \frac{2AB}{\rho}.$$

where $B > 1$ is the sufficiently large constant in Lemma 6.3. Thus Eq. (6.4) is now $O(\sqrt{\varepsilon'}+\tau^{1/9})$. (We remark that a different δ is used when proving Theorem 6.1.)

$\mathbf{E}[\tilde{\mathbf{I}}^{\mathbf{c}}_{\mathbf{S}_{\rho,\mathbf{t},\varepsilon'}}(\mathbf{M_p}(\mathbf{Y}))] = \mathbf{O}(\sqrt{\varepsilon'}+\tau^{\mathbf{1/9}})$: It suffices to show

$$\mathbf{E}[\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(M_p(Y))] \approx_\varepsilon \mathbf{E}[\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(M_p(X))].$$

We remark that $\tilde{I}^c_{S_{\rho,t,\varepsilon'}}$ can be assumed to be even in both x_1, x_2. If not, then consider the symmetrization

$$(\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(x_1,x_2,x_3,x_4)+\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(-x_1,x_2,x_3,x_4)+\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(x_1,-x_2,x_3,x_4)+\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(-x_1,-x_2,x_3,x_4))/4, \qquad (6.5)$$

which does not affect any of our properties (i),(ii), (iii).

Now, by our choice of k, δ and item (i), we have by Lemma 6.3 (with $\alpha = 2c$) that

$$|\mathbf{E}[\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(M_p(X))] - \mathbf{E}[\tilde{I}^c_{S_{\rho,t,\varepsilon'}}(M_p(Y))]| < \varepsilon'.$$

This completes our proof by applying Eq. (6.2) with $Z = Y$.

■

The following Corollary is proven similarly as Lemma 6.4, but uses anticoncentration under bounded independence (which we just proved in Lemma 6.5). The proof is in Section D.

Corollary 6.6. Let $\eta, \eta' \ge 0$ be given, and let $Y_1, \dots, Y_n$ be k-independent Bernoulli for k as in Lemma 6.5 with $\varepsilon' = \min\{\eta/\sqrt{\delta}, \eta'\}$. Also assume $k \ge \lceil 2/\delta \rceil$. Then

$$\mathbf{Pr}[|p(X)-t| \le \eta\cdot(\sqrt{p_1(X)}+\sqrt{p_2(X)}+1)+\eta'] = O(\sqrt{\eta'}+(\eta^2/\delta)^{1/4}+\tau^{1/9}+\exp(-\Omega(1/\delta))).$$

We are now ready to prove the main theorem of this section.

Proof (of Theorem 6.1). Consider the region $R \subset \mathbb{R}^4$ defined by $R = \{(x_1,x_2,x_3,x_4) : x_1^2 - x_2^2 + x_3 + x_4 + C + \Upsilon \ge 0\}$. Then note that $I_{[0,\infty)}(p(x)) = 1$ if and only if $I_R(M_p(x)) = 1$. It thus suffices to show that I_R is fooled in expectation by bounded independence.

We set $\rho = \varepsilon^4$, $c = 1/\rho$, and $1/\delta = 2Bc$ for B the constant in the statement of Lemma 6.3. We now show a chain of inequalities to give our theorem:

$$\mathbf{E}[I_R(M_p(X))] \approx_{\varepsilon+\tau^{1/9}} \mathbf{E}[\tilde{I}_R^c(M_p(X))] \approx_{\varepsilon} \mathbf{E}[\tilde{I}_R^c(M_p(Y))] \approx_{\varepsilon+\tau^{1/9}} \mathbf{E}[I_R(M_p(Y))]$$

$\mathbf{E}[\mathbf{I_R}(\mathbf{M_p}(\mathbf{X}))] \approx_{\varepsilon+\tau^{1/9}} \mathbf{E}[\tilde{\mathbf{I}}_{\mathbf{R}}^{\mathbf{c}}(\mathbf{M_p}(\mathbf{X}))]$: Similarly to as in the proof of Lemma 6.5,

$$d_2(x, \partial R) \geq \frac{1}{2} \cdot \min\left\{\frac{|x_1^2 - x_2^2 + x_3 + x_4 + C + \Upsilon|}{2(|x_1| + |x_2| + 1)}, \sqrt{|x_1^2 - x_2^2 + x_3 + x_4 + C + \Upsilon|}\right\},$$

and thus by Lemma 6.4,

$$\begin{aligned}\mathbf{Pr}[d_2(M_p(X), \partial R) \leq w] &\leq \mathbf{Pr}[|p(X)| \leq 4w \cdot (\sqrt{p_1(X)} + \sqrt{p_2(X)} + 1)] + \mathbf{Pr}[|p(X)| \leq 4w^2] \\ &= O(w + \sqrt{w} + (w^2/\delta)^{1/4} + \tau^{1/9} + \exp(-\Omega(1/\delta)))\end{aligned}$$

Now, noting $|\mathbf{E}[I_R(M_p(X))] - \mathbf{E}[\tilde{I}_R^c(M_p(X))]| \leq \mathbf{E}[|I_R(M_p(X))] - \tilde{I}_R^c(M_p(X))|]$ and applying Theorem 4.10,

$$\begin{aligned}&|\mathbf{E}[I_R(M_p(X))] - \mathbf{E}[\tilde{I}_R^c(M_p(X))]| \\ &\quad\leq \mathbf{Pr}[d_2(M_p(X), \partial R) \leq 2\rho] + O\left(\sum_{s=1}^{\infty} 2^{-2s} \cdot \mathbf{Pr}[2^s\rho < d_2(M_p(X), \partial R) \leq 2^{s+1}\rho]\right) \\ &\quad\leq O(\sqrt{\rho} + (\rho^2/\delta)^{1/4} + \tau^{1/9} + \exp(-\Omega(1/\delta)) \\ &\qquad + O\left(\sum_{s=1}^{\infty} 2^{-2s} \cdot (\sqrt{2^{s+1}\rho} + (2^{2s+2}\rho^2/\delta)^{1/4} + \tau^{1/9} + \exp(-\Omega(1/\delta)))\right) \\ &\quad= O(\varepsilon + \tau^{1/9})\end{aligned}$$

by choice of ρ, δ and applications of Lemma 6.4.

$\mathbf{E}[\tilde{\mathbf{I}}_{\mathbf{R}}^{\mathbf{c}}(\mathbf{M_p}(\mathbf{X}))] \approx_{\varepsilon} \mathbf{E}[\tilde{\mathbf{I}}_{\mathbf{R}}^{\mathbf{c}}(\mathbf{M_p}(\mathbf{Y}))]$: As in Eq. (6.5), we can assume $\tilde{I}_R^c$ is even in x_1, x_2. We apply Lemma 6.3 with $\alpha = 2c$, noting that $1/\delta = B\alpha$ and that our setting of k is sufficiently large.

$\mathbf{E}[\tilde{\mathbf{I}}_{\mathbf{R}}^{\mathbf{c}}(\mathbf{M_p}(\mathbf{Y}))] \approx_{\varepsilon+\tau^{1/9}} \mathbf{E}[\mathbf{I_R}(\mathbf{M_p}(\mathbf{Y}))]$: The argument is identical as with the first inequality, except that we use Corollary 6.6 instead of Lemma 6.4. We remark that we do have sufficient independence to apply Corollary 6.6 since, mimicking our analysis of the first inequality, we have

$$\begin{aligned}&\mathbf{Pr}[|p(Y)| \leq 4\rho \cdot (\sqrt{p_1(Y)} + \sqrt{p_2(Y)} + 1)] + \mathbf{Pr}[|p(Y)| \leq 4\rho^2] \\ &\quad\leq \mathbf{Pr}[|p(Y)| \leq 4\rho \cdot (\sqrt{p_1(Y)} + \sqrt{p_2(Y)} + 1)] + \mathbf{Pr}[|p(Y)| \leq \varepsilon^2]\end{aligned} \tag{6.6}$$

since $\rho^2 = o(\varepsilon^2)$ (we only changed the second summand). To apply Corollary 6.6 to Eq. (6.6), we need $k \geq \lceil 2/\delta \rceil$, which is true, and $k = \Omega(1/(\varepsilon'')^4)$, for $\varepsilon'' = \min\{\rho/\sqrt{\delta}, \varepsilon^2\} = \varepsilon^2$, which is also true. Corollary 6.6 then tells us Eq. (6.6) is $O(\varepsilon + \tau^{1/9})$.

■

Our main theorem of this Section (Theorem 6.1) also holds under the case that the X_i, Y_i are standard normal, and without any error term depending on τ. We give a proof in Section D.2, by reducing back to the Bernoulli case.

7 Reduction to the regular case

In this section, we complete the proof of Theorem 1.1. We accomplish this by providing a reduction from the general case to the regular case. In fact, such a reduction can be shown to hold for any degree $d \geq 1$ and establishes the following:

Theorem 7.1. Suppose K_d-wise independence ε-fools the class of τ-regular degree-d PTF's, for some parameter $0 < \tau \leq \varepsilon$. Then $(K_d + L_d)$-wise independence ε-fools all degree-d PTFs, where $L_d = (1/\tau) \cdot \big(d \log(1/\tau)\big)^{O(d)}$.

Noting that τ-regularity implies that the maximum influence of any particular variable is at most $d \cdot \tau$, Theorem 6.1 implies that degree-2 PTF's that are τ-regular, for $\tau = O(\varepsilon^9)$, are ε-fooled by K_2-wise independence for $K_2 = O(\varepsilon^{-8}) = \text{poly}(1/\varepsilon)$. By plugging in $\tau = O(\varepsilon^9)$ in the above theorem we obtain Theorem 1.1. The proof of Theorem 7.1 is based on recent machinery from [15][4]. Here we give a sketch, with full details in Section E.

Proof (Sketch). (of Theorem 7.1). Any boolean function f on $\{-1, 1\}^n$ can be expressed as a binary decision tree where each internal node is labeled by a variable, every root-to-leaf path corresponds to a restriction ρ that fixes the variables as they are set on the path, and every leaf is labeled with the restricted subfunction f_ρ. The main claim is that, if f is a degree-d PTF, then it has such a decision-tree representation with certain strong properties. In particular, given an arbitrary degree-d PTF $f = \text{sgn}(p)$, by [15] there exists a decision tree $\mathcal{T}$ of depth $(1/\tau) \cdot \big(d \log(1/\tau)\big)^{O(d)}$, so that with probability $1 - \tau$ over the choice of a random root-to-leaf path[5] ρ, the restricted subfunction (leaf) $f_\rho = \text{sgn}(p_\rho)$ is either a τ-regular degree-d PTF or τ-close to a constant function.

Our proof of Theorem 7.1 is based on the above structural lemma. Under the uniform distribution, there is some particular distribution on the leaves (the tree is not of uniform height); then conditioned on the restricted variables the variables still undetermined at the leaf are still uniform. With $(K_d + L_d)$-wise independence, a random walk down the tree arrives at each leaf with the same probability as in the uniform case (since the depth of the tree is at most L_d). Hence, the probability mass of the "bad" leaves is at most $\tau \leq \varepsilon$ even under bounded independence. Furthermore, the induced distribution on each leaf (over the unrestricted variables) is K_d-wise independent. Consider a good leaf. Either the leaf is τ-regular, in which case we can apply Theorem 6.1, or it is τ-close to a constant function. At this point though we arrive at a technical issue. The statement and proof in [15] concerning "close-to-constant" leaves holds only under the uniform distribution. For our result, we need a stronger statement that holds under any distribution (on the variables that do not appear in the path) that has sufficiently large independence. By simple modifications of the proof in [15], we show that the statement holds even under $O(d \cdot \log(1/\tau))$-wise independence. ■

8 Fooling intersections of threshold functions

Our approach also implies that the intersection of halfspaces (or even degree-2 threshold functions) is fooled by bounded independence. While Theorem D.1 implies that $\Omega(\varepsilon^{-8})$-wise independence fools GW rounding, we can do much better by noting that to fool GW rounding it suffices to fool the intersection of two halfspaces under the Gaussian measure.

This is because in the GW rounding scheme for MaxCut, each vertex u is first mapped to a vector x_u of unit norm, and the side of a bipartition u is placed in is decided by $\text{sgn}(\langle x_u, r\rangle)$

[4]We note that [30] uses a similar approach to obtain their PRG's for degree-d PTF's. Their methods are not directly applicable in our setting, one reason being that that their notion of "regularity" is different from ours.

[5]A "random root-to-leaf path" corresponds to the standard uniform random walk on the tree.

for a random Gaussian vector r. For a vertex u, let H_u^+ be the halfspace $\langle x_u, r\rangle > 0$, and let H_u^- be the halfspace $\langle -x_u, r\rangle > 0$. Then note that the edge (u, v) is cut if and only if $r \in (H_u^+ \cap H_v^-) \cup (H_u^- \cap H_v^+)$, i.e. r must be in the union of the intersection of two halfspaces. Thus if we define the region R^+ to be the topright quadrant of $\mathbb{R}^2$, and R^- to be the bottom left quadrant of $\mathbb{R}^2$, then we are interested in fooling

$$\mathbf{E}[I_{R^+\cup R^-}(\langle x_u, r\rangle, \langle -x_v, r\rangle)] = \mathbf{E}[I_{R^+}(\langle x_u, r\rangle, \langle -x_v, r\rangle)] + \mathbf{E}[I_{R^+}(\langle -x_u, r\rangle, \langle x_v, r\rangle)],$$

since the sum of such expectations over all edges (u, v) gives us the expected number of edges that are cut (note equality holds above since the two halfspace intersections are disjoint). The following theorem then implies that to achieve a maximum cut within a factor $.878... - \varepsilon$ of optimal in expectation, it suffices that the entries of the random normal vector r have entries that are $\Omega(1/\varepsilon^2)$-wise independent. The proof of the theorem is in Section F.

Theorem 8.1. Let $H_1 = \{x : \langle a, x\rangle > \theta_1\}$ and $H_2 = \{x : \langle b, x\rangle > \theta_2\}$ be two halfspaces, with $\|a\|_2 = \|b\|_2 = 1$. Let X, Y be n-dimensional vectors of standard normals with the X_i independent and the Y_i k-wise independent for $k = \Omega(1/\varepsilon^2)$. Then $|\mathbf{Pr}[X \in H_1 \cap H_2] - \mathbf{Pr}[Y \in H_1 \cap H_2]| < \varepsilon$.

The proof of Theorem 8.1 can be summarized in one sentence: FT-mollify the indicator function of $\{x : x_1 \geq \theta_1, x_2 \geq \theta_2\} \subset \mathbb{R}^2$. We also in Section F discuss how our proof of Theorem 8.1 easily generalizes to handle the intersection of m halfspaces, or even m degree-2 PTF's, for any constant m, as well as generalizations to case that X, Y are Bernoulli vectors as opposed to Gaussian. Our dependence on m in all cases is polynomial.

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A Basic linear algebra facts

In this subsection we record some basic linear algebraic facts used in our proofs.

We start with two elementary facts.

Fact A.1. If $A, P \in \mathbb{R}^{n\times n}$ with P invertible, then the eigenvalues of A and $P^{-1}AP$ are identical.

Fact A.2. For $A \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, and for integer $k > 0$, $\mathrm{tr}(A^k) = \sum_i \lambda_i^k$.

Note Fact A.1 and Fact A.2 imply the following.

Fact A.3. For a real matrix $A \in \mathbb{R}^{n\times n}$ and invertible matrix $P \in \mathbb{R}^{n\times n}$,

$$\|P^{-1}AP\|_2 = \|A\|_2.$$

The following standard result will be useful:

Theorem A.4 (Spectral Theorem [41, Section 6.4]). If $A \in \mathbb{R}^{n\times n}$ is symmetric, there exists an orthogonal $Q \in \mathbb{R}^{n\times n}$ with $\Lambda = Q^T AQ$ diagonal. In particular, all eigenvalues of A are real.

Definition A.5. For a real symmetric matrix A, we define $\lambda_{\min}(A)$ to be the smallest magnitude of a non-zero eigenvalue of A (in the case that all eigenvalues are 0, we set $\lambda_{\min}(A) = 0$). We define $\|A\|_\infty$ to be the largest magnitude of an eigenvalue of A.

We now give a simple lemma that gives an upper bound on the magnitude of the trace of a symmetric matrix with positive eigenvalues.

Lemma A.6. Let $A \in \mathbb{R}^{n\times n}$ be symmetric with $\lambda_{\min}(A) > 0$. Then $|\mathrm{tr}(A)| \le \|A\|_2^2/\lambda_{\min}(A)$.

Proof. We have

$$\begin{aligned}
|\mathrm{tr}(A)| &= \left|\sum_{i=1}^{n} \lambda_i\right| \\
&\le \frac{\|A\|_2}{\lambda_{\min}(A)} \cdot \sqrt{\sum_{i=1}^{n} \lambda_i^2} \\
&= \frac{\|A\|_2^2}{\lambda_{\min}(A)}
\end{aligned}$$

We note $\sum_{i=1}^n \lambda_i^2 = \|A\|_2^2$, implying the final equality. Also, there are at most $\|A\|_2^2/(\lambda_{\min}(A))^2$ non-zero λ_i. The sole inequality then follows by Cauchy-Schwarz. ■

B Useful facts about polynomials

B.1 Facts about low-degree polynomials. We view $\{-1,1\}^n$ as a probability space endowed with the uniform probability measure. For a function $f : \{-1,1\}^n \to \mathbb{R}$ and $r \ge 1$, we let $\|f\|_r$ denote $(\mathbf{E}_x[|f(x)|^r])^{1/r}$.

Our first fact is a consequence of the well-known hypercontractivity theorem.

Theorem B.1 (Hypercontractivity [5, 8]). If f is a degree-d polynomial and $1 \le r < q \le \infty$,

$$\|f\|_q \le \sqrt{\frac{q-1}{r-1}}^{\,d} \|f\|_r.$$

Our second fact is an anticoncentration theorem for low-degree polynomials over independent standard Gaussian random variables.

Theorem B.2 (Gaussian Anticoncentration [12]). For f a non-zero, n-variate, degree-d polynomial,

$$\mathbf{Pr}[|f(G_1,\ldots,G_n)-t| \le \varepsilon \cdot \mathbf{Var}[f]] = O(d\varepsilon^{1/d})$$

for all $\varepsilon \in (0,1)$ and $t \in \mathbb{R}$. Here $G_1,\ldots,G_n \sim \mathcal{N}(0,1)$ are independent. (Here, and henceforth, $\mathcal{N}(\mu,\sigma^2)$ denotes the Gaussian distribution with mean μ and variance σ^2.)

The following is a statement of the Invariance Principle of Mossell, O'Donnell, and Oleszkiewicz [32], in the special case when the random variables X_i are Bernoulli.

Theorem B.3 (Invariance Principle [32]). Let $X_1,\ldots,X_n$ be independent ± 1 Bernoulli, and let p be a degree-d multilinear polynomial with $\sum_{|S|>0} \widehat{p}_S^2 = 1$ and $\max_i \mathrm{Inf}_i(p) \le \tau$. Then

$$\sup_t |\mathbf{Pr}[p(X_1,\ldots,X_n) \le t] - \mathbf{Pr}[p(G_1,\ldots,G_n) \le t]| = O(d\tau^{1/(4d+1)})$$

where the $G_i \sim \mathcal{N}(0,1)$ are independent.

The following tail bound argument is standard (see for example [3]). We repeat the argument here just to point out that only bounded independence is required.

Theorem B.4 (Tail bound). If f is a degree-d polynomial, $t > 8^{d/2}$, and X is drawn at random from a $(dt^{2/d})$-wise independent distribution over $\{-1,1\}^n$, then

$$\mathbf{Pr}[|f(X)| \ge t\|f\|_2] = \exp(-\Omega(dt^{2/d})).$$

Proof. Suppose $k > 2$. By Theorem B.1,

$$\mathbf{E}[|f(X)|^k] \le k^{dk/2} \cdot \|f\|_2^k,$$

implying

$$\mathbf{Pr}[|f(X)| \ge t\|f\|_2] \le (k^{d/2}/t)^k \tag{B.1}$$

by Markov's inequality. Set $k = 2 \cdot \lfloor t^{2/d}/4 \rfloor$ and note $k > 2$ as long as $t > 8^{d/2}$. Now the right hand side of Eq. (B.1) is at most $2^{-dk/2}$, as desired. Finally, note independence was only used to bound $\mathbf{E}[|f(X)|^k]$, which for k even equals $\mathbf{E}[f(X)^k]$ and is thus determined by dk-independence. ■

B.2 Facts about quadratic forms. The following facts are concerned with quadratic forms, i.e. polynomials $p(x) = \sum_{i \le j} a_{i,j} x_i x_j$. We often represent a quadratic form p by its associated symmetric matrix A_p, where

$$(A_p)_{i,j} = \begin{cases} a_{i,j}/2, & i < j \\ a_{j,i}/2, & i > j \\ a_{i,j}, & i = j \end{cases}$$

so that $p(x) = x^T A_p x$.

The following is a bound on moments for quadratic forms.

Lemma B.5. Let $f(x)$ be a degree-2 polynomial. Then, for $X = (X_1,\ldots,X_n)$ a vector of independent Bernoullis,

$$\mathbf{E}[|f(X)|^k] \le 2^{O(k)}(\|A_f\|_2 k^k + |\mathrm{tr}(A_f)|^k).$$

Proof. Over the hypercube we can write $f = q + \mathrm{tr}(A_f)$ where q is multilinear. Note $\|A_q\|_2 \leq \|A_f\|_2$. Then by Theorem B.1,

$$\begin{aligned}
\mathbf{E}[|f(x)|^k] &= \mathbf{E}[|q(x) + \mathrm{tr}(A_f)|^k] \\
&\leq \sum_{i=0}^{k} (\|A_f\|_2 \cdot i)^i |\mathrm{tr}(A_f)|^{k-i} \\
&\leq \sum_{i=0}^{k} (\|A_f\|_2 \cdot k)^i |\mathrm{tr}(A_f)|^{k-i} \\
&= 2^{O(k)} \max\{\|A_f\|_2 \cdot k, |\mathrm{tr}(A_f)|\}^k
\end{aligned}$$

■

The following corollary now follows from Theorem B.4 and Lemma A.6.

Corollary B.6. Let f be a quadratic form with A_f positive semidefinite, $\|A_f\|_2 \leq 1$, and $\lambda_{\min}(A_f) \geq \delta$ for some $\delta \in (0, 1]$. Then, for x chosen at random from a $\lceil 2/\delta \rceil$-independent family over $\{-1, 1\}^n$,

$$\mathbf{Pr}[f(x) > 2/\delta] = \exp(-\Omega(1/\delta)).$$

Proof. Write $f = g + C$ via Lemma A.6 with $0 \leq C \leq 1/\delta$ and g multilinear, $\|A_g\|_2 \leq \|A_f\|_2 \leq 1$. Apply Theorem B.4 to g with $t = 1/\delta$. ■

The following lemma gives a decomposition of any multi-linear quadratic form as a sum of quadratic forms with special properties for the associated matrices. It is used in the proof of Theorem 6.1.

Lemma B.7. Let $\delta > 0$ be given. Let f be a multilinear quadratic form. Then f can be written as $f_1 - f_2 + f_3$ for quadratic forms f_1, f_2, f_3 where:

1. A_{f_1}, A_{f_2} are positive semidefinite with $\lambda_{\min}(A_{f_1}), \lambda_{\min}(A_{f_2}) \geq \delta$.
2. $\|A_{f_3}\|_\infty < \delta$.
3. $\|A_{f_1}\|_2, \|A_{f_2}\|_2, \|A_{f_3}\|_2 \leq \|A_f\|_2$.

Proof. Since A_f is real and symmetric, we can find an orthogonal matrix Q such that $\Lambda = Q^T A_f Q$ is diagonal. Each diagonal entry of Λ is either at least δ, at most $-\delta$, or in between. We create a matrix P containing all entries of Λ which are at least δ, with the others zeroed out. We similarly create N to have all entries at most $-\delta$. We place the remaining entries in R. We then set $A_{f_1} = QPQ^T, A_{f_2} = QNQ^T, A_{f_3} = QRQ^T$. Note $\|\Lambda\|_2^2 = \|A_f\|_2^2$ by Fact A.3, so since we remove terms from Λ form each A_{f_i}, their Frobenius norms can only shrink. The eigenvalue bounds hold by construction and Fact A.1. ■

C Why the previous approaches failed

In this section, we attempt to provide an explanation as to why the approaches of [14] and [25] fail to fool degree-2 PTFs.

C.1 Why the approximation theory approach failed The analysis in [14] crucially exploits the strong concentration and anti-concentration properties of the gaussian distribution. (Recall that in the linear regular case, the random variable $\langle w, x\rangle$ is approximately Gaussian.) Now consider a regular degree-2 polynomial p and the corresponding PTF $f = \mathrm{sgn}(p)$. Since p is regular, it has still has "good" concentration and anti-concentration properties – though quantitatively inferior than those of the Gaussian. Hence, one would hope to argue as follows: use the univariate polynomial P (constructed using approximation theory), allowing its degree to increase if necessary, and carry out the analysis of the error as in the linear case.

The reason this fails is because the (tight) concentration properties of p – as implied by hypercontractivity – are not sufficient for the analysis to bound the error of the approximation, even if we let the degree of the polynomial P tend to infinity. (Paradoxically, the error coming from the worst-case analysis becomes worse as the degree of P increases.)

Without going into further details, we mention that an additional problem for univariate approximations to work is this: the (tight) anti-concentration properties of p – obtained via the Invariance Principle and the anti-concentration bounds of [12] – are quantitatively weaker than what is required to bound the error, even in the region where P has small point-wise error (from the sgn function).

C.2 Why the analysis for univariate FT-mollification failed We discuss why the argument in [25] failed to generalize to higher degree. Recall that the argument was via the following chain of inequalities:

$$\mathbf{E}[I_{[0,\infty)}(p(X))] \approx_\varepsilon \mathbf{E}[\tilde{I}^c_{[0,\infty)}(p(X))] \approx_\varepsilon \mathbf{E}[\tilde{I}^c_{[0,\infty)}(p(Y))] \approx_\varepsilon \mathbf{E}[I_{[0,\infty)}(p(Y))] \tag{C.1}$$

The step that fails for high-degree PTFs is the second inequality in Eq. (C.1), which was argued by Taylor's theorem. Our bounds on derivatives of $\tilde{I}^c_{[0,\infty)}$, the FT-mollification of $I_{[0,\infty)}$ for a certain parameter $c = c(\varepsilon)$ to make sure $|I_{[0,\infty)} - \tilde{I}^c_{[0,\infty)}| < \varepsilon$ "almost everywhere", are such that $||(\tilde{I}^c_{[0,\infty)})^{(k)}||_\infty \geq 1$ for all k. Thus, we have that the error term from Taylor's theorem is at least $\mathbf{E}[(p(X))^k]/k!$. The problem comes from the numerator. Since we can assume the sum of squared coefficients of p is 1 (note the sgn function is invariant to scaling of its argument), known (and tight) moment bounds (via hypercontractivity) only give us an upper bound on $\mathbf{E}[(p(x))^k]$ which is larger than $k^{dk/2}$, where degree$(p) = d$. Thus, the error from Taylor's theorem does not decrease to zero by increasing k for $d \geq 2$, since we only are able to divide by $k! \leq k^k$ (in fact, strangely, increasing the amount of independence k *worsens* this bound).

D Proofs omitted from Section 6

D.1 Boolean setting. We next give a proof of Lemma 6.4, where p_1, p_2, δ are as in Section 6 (recall $p = p_1 - p_2 + p_3 + p_4 + C$ where p_1, p_2 are positive semidefinite with minimum non-zero eigenvalues at least δ).

Lemma 6.4 (restatement). *Let $\eta, \eta' \geq 0, t \in \mathbb{R}$ be given, and let $X_1, \ldots, X_n$ be independent Bernoulli. Then*

$$\mathbf{Pr}[|p(X) - t| \leq \eta \cdot (\sqrt{p_1(X)} + \sqrt{p_2(X)} + 1) + \eta'] = O(\sqrt{\eta'} + (\eta^2/\delta)^{1/4} + \tau^{1/9} + \exp(-\Omega(1/\delta))).$$

Proof. Applying Corollary B.6, we have

$$\mathbf{Pr}[\sqrt{p_1(X)} \geq \sqrt{2/\delta}] = \exp(-\Omega(1/\delta)),$$

and similarly for $\sqrt{p_2(X)}$. We can thus bound our desired probability by

$$\mathbf{Pr}[|p(X) - t| \leq 2\eta\sqrt{2/\delta} + \eta + \eta'] + \exp(-\Omega(1/\delta)).$$

By Theorem B.2, together with Theorem B.3, we can bound the probability in the lemma statement by

$$O(\sqrt{\eta'} + (\eta^2/\delta)^{1/4} + \tau^{1/9} + \exp(-\Omega(1/\delta))).$$

■

Corollary 6.6 (restatement). *Let $\eta, \eta' \geq 0$ be given, and let $Y_1, \ldots, Y_n$ be k-independent Bernoulli for k as in Lemma 6.5 with $\varepsilon' = \min\{\eta/\sqrt{\delta}, \eta'\}$. Also assume $k \geq \lceil 2/\delta \rceil$. Then*

$$\mathbf{Pr}[|p(X) - t| \leq \eta \cdot (\sqrt{p_1(X)} + \sqrt{p_2(X)} + 1) + \eta'] = O(\sqrt{\eta'} + (\eta^2/\delta)^{1/4} + \tau^{1/9} + \exp(-\Omega(1/\delta))).$$

Proof. There were two steps in the proof of Lemma 6.4 which required using the independence of the X_i. The first was in the application of Corollary B.6, but that only required $\lceil 2/\delta \rceil$-wise independence, which is satisfied here. The next was in using the anticoncentration of $p(X)$ (the fact that $\mathbf{Pr}[|p(X) - t| < s] = O(\sqrt{s} + \tau^{1/9})$ for any $t \in \mathbb{R}$ and $s > 0$). However, given Lemma 6.5, anticoncentration still holds under k-independence. ■

D.2 Gaussian Setting In the following Theorem we show that the conclusion of Theorem 6.1 holds even under the Gaussian measure.

Theorem D.1. Let $0 < \varepsilon < 1$ be given. Let $G = (G_1, \ldots, G_n)$ be a vector of independent standard normal random variables, and $G' = (G'_1, \ldots, G'_n)$ be a vector of $2k$-wise independent standard normal random variables for k a sufficiently large multiple of $1/\varepsilon^8$. If $p(x) = \sum_{i \leq j} a_{i,j} x_i x_j$ has $\sum_{i \leq j} a_{i,j}^2 = 1$,

$$\mathbf{E}[\mathrm{sgn}(p(G))] - \mathbf{E}[\mathrm{sgn}(p(G'))] = O(\varepsilon).$$

Proof. Our proof is by a reduction to the Bernoulli case, followed by an application of Theorem 6.1. We replace each G_i with $Z_i = \sum_{j=1}^{N} X_{i,j}/\sqrt{N}$ for a sufficiently large N to be determined later. We also replace each G'_i with $Z'_i = \sum_{j=1}^{N} Y_{i,j}/\sqrt{N}$. We determine these $X_{i,j}, Y_{i,j}$ as follows. Let $\Phi : \mathbb{R} \to [0, 1]$ be the cumulative distribution function (CDF) of the standard normal. Define $T_{-1,N} = -\infty$, $T_{N,N} = \infty$, and $T_{k,N} = \Phi^{-1}(2^{-N} \sum_{j=0}^{k} \binom{N}{k})$ for $0 \leq k \leq N$. Now, after a G_i is chosen according to a standard normal distribution, we identify the unique k_i such that $T_{k_i - 1,N} \leq G_i < T_{k_i,N}$. We then randomly select a subset of k_i of the $X_{i,j}$ to make 1, and we set the others to -1. The $Y_{i,j}$ are defined similarly. It should be noted that the $X_{i,j}, Y_{i,j}$ are Bernoulli random variables, with the $X_{i,j}$ being independent and the $Y_{i,j}$ being $2k$-wise independent. Furthermore, we define the nN-variate polynomial $p' : \{-1, 1\}^{nN} \to \mathbb{R}$ to be the one obtained from this procedure, so that $p(G) = p'(X)$. We then define $p''(x) = \alpha \cdot p'(x)$ for $\alpha = (\sum_{i<j} a_{i,j}^2 + (1 - 1/N) \sum_i a_{i,i}^2)^{-1}$ so that the sum of squared coefficients in p'' (ignoring constant terms, some of which arise because the $x_{i,j}^2$ terms are 1 on the hypercube) is 1. It should be observed that $1 \leq \alpha \leq 1 + 1/(N - 1)$.

Now, we make the setting $\epsilon = \log^{1/3}(N)/\sqrt{N}$. By the Chernoff bound,

$$\mathbf{Pr}[|k_i - N/2| \geq \epsilon N/2] = o(1) \text{ as } N \text{ grows.} \tag{D.1}$$

Claim D.2. If $(1 - \epsilon)N/2 \leq k_i \leq (1 + \epsilon)N/2$, then $|T_{k_i,N} - T_{k_i+1,N}| = o(1)$.

Before proving the claim, we show how now we can use it to prove our Theorem. We argue by the following chain of inequalities:

$$\mathbf{E}[\mathrm{sgn}(p(G))] \approx_\varepsilon \mathbf{E}[\mathrm{sgn}(p''(X))] \approx_\varepsilon \mathbf{E}[\mathrm{sgn}(p''(Y))] \approx_\varepsilon \mathbf{E}[\mathrm{sgn}(p(G'))].$$

$\mathbf{E}[\mathrm{sgn}(\mathbf{p}(\mathbf{G}))] \approx_\varepsilon \mathbf{E}[\mathrm{sgn}(\mathbf{p}''(\mathbf{X}))]$: First we condition on the event $\mathcal{E}$ that $|Z_i - G_i| \leq \varepsilon^3/n^2$ for all $i \in [n]$; this happens with probability $1-o(1)$ as N grows by coupling Claim D.2 and Eq. (D.1), and applying a union bound over all $i \in [n]$. We also condition on the event $\mathcal{E}'$ that $|G_i| = O(\sqrt{\log(n/\varepsilon)})$ for all $i \in [n]$, which happens with probability $1-\varepsilon^2$ by a union bound over $i \in [n]$ since a standard normal random variable has probability $e^{-\Omega(x^2)}$ of being larger than x in absolute value. Now, conditioned on $\mathcal{E}, \mathcal{E}'$, we have

$$|p(G)-p''(X)| \leq n^2(\varepsilon^3/n^2)^2+(\varepsilon^3/n^2)\sum_i |G_i| \left(\sum_j |a_{i,j}|\right) \leq \varepsilon^2+(\varepsilon^3/n^2)\cdot O(\sqrt{\log(n/\varepsilon)})\cdot \sum_{i,j} |a_{i,j}|.$$

We note $\sum_{i,j} a_{i,j}^2 = 1$, and thus $\sum_{i,j} |a_{i,j}| \leq n$ by Cauchy-Schwarz. We thus have that $|p'(X) - p(G)| \leq \varepsilon^2$ with probability at least $1-\varepsilon^2$, and thus $|p''(X) - p(G)| \leq \varepsilon^2 + |(\alpha - 1)\cdot p(X)|$ with probability at least $1-\varepsilon^2$. We finally condition on the event $\mathcal{E}''$ that $|(\alpha-1)\cdot p'(X)| \leq \varepsilon^2$. Since p' can be written as a multilinear quadratic form with sum of squared coefficients at most 1, plus its trace $\mathrm{tr}(A_{p'})$ (which is $\sum_i a_{i,i} \leq \sqrt{n}$, by Cauchy-Schwarz), we have

$$\mathbf{Pr}[|(\alpha-1)\cdot p'(X)| \geq \varepsilon^2] \leq \mathbf{Pr}[|p'(X)| \geq \varepsilon^2\cdot(N-1)] = o(1),$$

which for large enough N and the fact that $\|p'\|_2 = O(1+\mathrm{tr}(A_{p'}))$ irrespective of N, is at most

$$\mathbf{Pr}[|p'(X)| \geq c\cdot \log(1/\varepsilon)\|p'\|_2],$$

for a constant c we can make arbitrarily large by increasing N. We thus have $\mathbf{Pr}[\mathcal{E}''] \geq 1-\varepsilon^2$ by Theorem B.4. Now, conditioned on $\mathcal{E} \wedge \mathcal{E}' \wedge \mathcal{E}''$, $\mathrm{sgn}(p''(X)) \neq \mathrm{sgn}(p(G))$ can only occur if $|p''(X)| = O(\varepsilon^2)$. However, by anticoncentration (Theorem B.2) and the Invariance Principle (Theorem B.3), this occurs with probability $O(\varepsilon)$ for N sufficiently large (note the maximum influence of p'' goes to 0 as $N \to \infty$).

$\mathbf{E}[\mathrm{sgn}(\mathbf{p}''(\mathbf{X}))] \approx_\varepsilon \mathbf{E}[\mathrm{sgn}(\mathbf{p}''(\mathbf{Y}))]$: Since the maximum influence τ of any $x_{i,j}$ in p'' approaches 0 as $N \to \infty$, we can apply Theorem 6.1 for N sufficiently large (and thus τ sufficiently small).

$\mathbf{E}[\mathrm{sgn}(\mathbf{p}''(\mathbf{Y}))] \approx_\varepsilon \mathbf{E}[\mathrm{sgn}(\mathbf{p}(\mathbf{G}'))]$: This case is argued identically as in the first inequality, except that we use anticoncentration of $p''(Y)$, which follows from Lemma 6.5, and we should ensure that we have sufficient independence to apply Theorem B.4 with $t = O(\log(1/\varepsilon))$, which we do.

Proof (of Claim D.2). The claim is argued by showing that for k_i sufficiently close to its expectation (which is $N/2$), the density function of the Gaussian (i.e. the derivative of its CDF) is sufficiently large that the distance we must move from $T_{k_i,N}$ to $T_{k_i+1,N}$ to change the CDF by $\Theta(1/\sqrt{N}) \geq 2^{-N}\binom{N}{k_i+1}$ is small. We argue the case $(1-\epsilon)N/2 \leq k_i \leq N/2$ since the case $N/2 \leq k_i \leq (1+\epsilon)N/2$ is argued symmetrically. Also, we consider only the case $k_i = (1-\epsilon)N/2$ exactly, since the magnitude of the standard normal density function is smallest in this case.

Observe that each Z_i is a degree-1 polynomial in the $X_{i,j}$ with maximum influence $1/N$, and thus by the Berry-Esséen Theorem,

$$\sup_{t\in\mathbb{R}} |\mathbf{Pr}[Z_i \le t] - \mathbf{Pr}[G_i \le t]| \le \frac{1}{\sqrt{N}}.$$

Also note that

$$\mathbf{Pr}[G_i \le T_{k_i,N}] = \mathbf{Pr}\left[Z_i \le \frac{2k_i}{\sqrt{N}} - \sqrt{N}\right]$$

by construction. We thus have

$$\begin{aligned}\mathbf{Pr}[G_i \le T_{k_i,N}] &= \mathbf{Pr}\left[G_i \le \frac{2k_i}{\sqrt{N}} - \sqrt{N}\right] \pm \frac{1}{\sqrt{N}} \\ &= \mathbf{Pr}[G_i \le \log^{1/3}(N)] \pm \frac{1}{\sqrt{N}}\end{aligned}$$

By a similar argument we also have

$$\mathbf{Pr}[G_i \le T_{k_i+1,N}] = \mathbf{Pr}\left[G_i \le \log^{1/3}(N) + \frac{2}{\sqrt{N}}\right] \pm \frac{1}{\sqrt{N}}$$

Note though for $t = \Theta(\log^{1/3}(N))$, the density function f of the standard normal satisfies $f(t) = e^{-t^2/2} = N^{-o(1)}$. Thus, in this regime we can change the CDF by $\Theta(1/\sqrt{N})$ by moving only $N^{o(1)}/\sqrt{N} = o(1)$ along the real axis, implying $T_{k_i+1,N} - T_{k_i,N} = o(1)$. ■

■

E Proofs from Section 7

E.1 Proof of Theorem 7.1 We begin by stating the following structural lemma:

Theorem E.1. Let $f(x) = \mathrm{sgn}(p(x))$ be any degree-d PTF. Fix any $\tau > 0$. Then f is equivalent to a decision tree $\mathcal{T}$ of depth $\mathrm{depth}(d,\tau) \stackrel{\mathrm{def}}{=} (1/\tau)\cdot(d\log(1/\tau))^{O(d)}$ with variables at the internal nodes and a degree-d PTF $f_\rho = \mathrm{sgn}(p_\rho)$ at each leaf ρ, with the following property: with probability at least $1-\tau$, a random path from the root reaches a leaf ρ such that either: (i) f_ρ is τ-regular degree-d PTF, or (ii) For any $O(d\cdot\log(1/\tau))$-independent distribution $\mathcal{D}'$ over $\{-1,1\}^{n-|\rho|}$ there exists $b \in \{-1,1\}$ such that $\mathbf{Pr}_{x\sim\mathcal{D}'}[f_\rho(x) \neq b] \le \tau$.

We now prove Theorem 7.1 assuming Theorem E.1. We will need some notation. Consider a leaf of the tree $\mathcal{T}$. We will denote by ρ both the set of variables that appear on the corresponding root-to-leaf path and the corresponding partial assignment; the distinction will be clear from context. Let $|\rho|$ be the number of variables on the path. We identify a leaf ρ with the corresponding restricted subfunction $f_\rho = \mathrm{sgn}(p_\rho)$. We call a leaf "good" if it corresponds to either a τ-regular PTF or to a "close-to constant" function. We call a leaf "bad" otherwise. We denote by $L(\mathcal{T})$, $GL(\mathcal{T})$, $BL(\mathcal{T})$ the sets of leaves, good leaves and bad leaves of $\mathcal{T}$ respectively.

In the course of the proof we make repeated use of the following standard fact:

Fact E.2. Let $\mathcal{D}$ be a k-wise independent distribution over $\{-1,1\}^n$. Condition on any fixed values for any $t \le k$ bits of $\mathcal{D}$, and let $\mathcal{D}'$ be the projection of $\mathcal{D}$ on the other $n-t$ bits. Then $\mathcal{D}'$ is $(k-t)$-wise independent.

Throughout the proof, $\mathcal{D}$ denotes a $(K_d + L_d)$-wise independent distribution over $\{-1,1\}^n$. Consider a random walk on the tree $\mathcal{T}$. Let $LD(\mathcal{T},\mathcal{D})$ (resp. $LD(\mathcal{T},\mathcal{U})$) be the leaf that the random walk will reach when the inputs are drawn from the distribution $\mathcal{D}$ (resp. the uniform distribution). The following straightforward lemma quantifies the intuition that these distributions are the same. This holds because the tree has small depth and $\mathcal{D}$ has sufficient independence.

Lemma E.3. For any leaf $\rho \in L(\mathcal{T})$ we have $\mathbf{Pr}\big[LD(\mathcal{T},\mathcal{D}) = \rho\big] = \mathbf{Pr}\big[LD(\mathcal{T},\mathcal{U}) = \rho\big]$.

The following lemma says that, if ρ is a good leaf, the distribution induced by $\mathcal{D}$ on ρ $O(\varepsilon)$-fools the restricted subfunction f_ρ.

Lemma E.4. Let $\rho \in GL(\mathcal{T})$ be a good leaf and consider the projection $\mathcal{D}_{[n]\setminus\rho}$ of $\mathcal{D}$ on the variables not in ρ. Then we have $\left|\mathbf{Pr}_{x\sim\mathcal{D}_{[n]\setminus\rho}}[f_\rho(x) = 1] - \mathbf{Pr}_{y\sim\mathcal{U}_{[n]\setminus\rho}}[f_\rho(y) = 1]\right| \leq 2\varepsilon$.

Proof. If f_ρ is τ-regular, by Fact E.2 and recalling that $|\rho| \leq \text{depth}(d,\tau) \leq L_d$, the distribution $\mathcal{D}_{[n]\setminus\rho}$ is K_d-wise independent. Hence, the statement follows by assumption. Otherwise, f_ρ is ε-close to a constant, i.e. there exists $b \in \{-1,1\}$ so that for any $t = O(d\log(1/\tau))$-wise distribution $\mathcal{D}'$ over $\{-1,1\}^{n-|\rho|}$ we have $\mathbf{Pr}_{x\sim\mathcal{D}'}[f_\rho(x) \neq b] \leq \tau$ $(*)$. Since $L_d >> t$, Fact E.2 implies that $(*)$ holds both under $\mathcal{D}_{[n]\setminus\rho}$ and $\mathcal{U}_{[n]\setminus\rho}$, hence the statement follows in this case also, recalling that $\tau \leq \varepsilon$. ■

The proof of Theorem 7.1 now follows by a simple averaging argument. By the decision-tree decomposition of Theorem E.1, we can write

$$\mathbf{Pr}_{x\sim\mathcal{D}'_n}[f(x) = 1] = \sum_{\rho\in L(T)} \mathbf{Pr}\big[LD(\mathcal{T},\mathcal{D}') = \rho\big] \cdot \mathbf{Pr}_{y\in\mathcal{D}'_{[n]\setminus\rho}}\big[f_\rho(y) = 1\big]$$

where $\mathcal{D}'$ is either $\mathcal{D}$ or the uniform distribution $\mathcal{U}$. By Theorem E.1 and Lemma E.3 it follows that the probability mass of the bad leaves is at most ε under both distributions. Therefore, by Lemma E.3 and Lemma E.4 we get

$$\Big|\mathbf{Pr}_{x\sim\mathcal{D}}[f(x) = 1] - \mathbf{Pr}_{x\sim\mathcal{U}}[f(x) = 1]\Big| \leq \varepsilon +$$
$$\sum_{\rho\in GL(T)} \mathbf{Pr}\big[LD(\mathcal{T},\mathcal{U}) = \rho\big] \cdot \left|\mathbf{Pr}_{y\in\mathcal{U}_{[n]\setminus\rho}}\big[f_\rho(y) = 1\big] - \mathbf{Pr}_{y\in\mathcal{D}_{[n]\setminus\rho}}\big[f_\rho(y) = 1\big]\right| \leq 3\varepsilon.$$

This completes the proof of Theorem 7.1.

E.2 Proof of Theorem E.1 In this section we provide the proof of Theorem E.1. For the sake of completeness, we give below the relevant machinery from [15]. We note that over the hypercube every polynomial can be assumed to be multilinear, and so whenever we discuss a polynomial in this section it should be assumed to be multilinear. We start by defining the notion of the critical index of a polynomial:

Definition E.5 (critical index). Let $p : \{-1,1\}^n \to \mathbb{R}$ and $\tau > 0$. Assume the variables are ordered such that $\text{Inf}_i(p) \geq \text{Inf}_{i+1}(p)$ for all $i \in [n-1]$. The *τ-critical index* of p is the least i such that:

$$\frac{\text{Inf}_{i+1}(p)}{\sum_{j=i+1}^{n} \text{Inf}_j(p)} \leq \tau. \tag{E.1}$$

If Eq. (E.1) does not hold for any i we say that the τ-critical index of p is $+\infty$. If p is has τ-critical index 0, we say that p is *τ-regular*.

We will be concerned with polynomials p of degree-d. The work in [15] establishes useful random restriction lemmas for low-degree polynomials. Roughly, they are as follows: Let p be a degree-d polynomial. If the τ-critical index of p is zero, then $f = \mathrm{sgn}(p)$ is τ-regular and there is nothing to prove.

- If the τ-critical index of p is "very large", then a random restriction of "few" variables causes $f = \mathrm{sgn}(p)$ to become a "close-to-constant" function with probability $1/2^{O(d)}$. We stress that the distance between functions is measured in [15] with respect to the uniform distribution on inputs. As previously mentioned, we extend this statement to hold for any distribution with sufficiently large independence.

- If the τ-critical index of p is positive but not "very large", then a random restriction of a "small" number of variables – the variables with largest influence in p – causes p to become "sufficiently" regular with probability $1/2^{O(d)}$.

Formally, we require the following lemma which is a strengthening of Lemma 10 in [15]:

Lemma E.6. Let $p : \{-1,1\}^n \to \mathbb{R}$ be a degree-d polynomial and assume that its variables are in order of non-increasing influence. Let $0 < \tau', \beta < 1/2$ be parameters. Fix $\alpha = \Theta(d \log\log(1/\beta) + d \log d)$ and $\tau'' = \tau' \cdot (C' d \ln d \ln(1/\tau'))^d$, where C' is a universal constant. One of the following statements holds true:

1. The function $f = \mathrm{sgn}(p)$ is τ'-regular.

2. With probability at least $1/2^{O(d)}$ over a random restriction ρ fixing the first $L' = \alpha/\tau'$ variables of p, the function $f_\rho = \mathrm{sgn}(p_\rho)$ is β-close to a constant function. In particular, under any $O(d\log(1/\beta))$-wise independent distribution $\mathcal{D}'$ there exists $b \in \{-1,1\}$ such that $\mathbf{Pr}_{x \sim \mathcal{D}'}[f_\rho(x) \neq b] \leq \tau'$.

3. There exists a value $k \leq \alpha/\tau'$, such that with probability at least $1/2^{O(d)}$ over a random restriction ρ fixing the first k variables of p, the polynomial p_ρ is τ''-regular.

By applying the above lemma in a recursive manner we obtain Theorem E.1. This is done exactly as in the proof of Theorem 1 in [15]. We remark that in every recursive application of the lemma, the value of the parameter β is set to τ. This explains why $O(d\log(1/\tau))$-independence suffices in the second statement of Theorem E.1. Hence, to complete the proof of Theorem E.1, it suffices to establish Lemma E.6.

Proof (of Lemma E.6). We now sketch the proof of the lemma. The first statement of the lemma corresponds to the case that the value ℓ of τ'-critical index is 0, the second to the case that it is $\ell > L'$ and the third to $1 \leq \ell \leq L'$.

The proof of the second statement proceeds in two steps. Let H denote the first L' most influential variables of p and $T = [n] \setminus H$. Let $p'(x_H) = \sum_{S \subseteq H} \widehat{p}(S) x_S$. We first argue that with probability at least $2^{-\Omega(d)}$ over a random restriction ρ to H, the restricted polynomial $p_\rho(x_T)$ will have a "large" constant term $\widehat{p}_\rho(\emptyset) = p'(\rho)$, in particular at least $\theta = 2^{-\Omega(d)}$. The proof is based on the fact that, since the critical index is large, almost all of the Fourier weight of the polynomial p lies in p', and it makes use of a certain anti-concentration property over the hypercube. Since the randomness is over H and the projection of $\mathcal{D}$ on those variables is still uniform, the argument holds unchanged under $\mathcal{D}$.

In the second step, by an application of a concentration bound, we show that for at least half of these restrictions to H the surviving (non-constant) coefficients of p_ρ, i.e. the Fourier coefficients of the polynomial $p_\rho(x_T) - p'(\rho)$, have small ℓ_2 norm; in particular, we get that $\|p_\rho - p'_\rho\|_2 \leq \log(1/\beta)^{-d}$. We call such restrictions good. Since the projection of $\mathcal{D}$ on these "head" variables is uniform, the concentration bound applies as is.

Finally, we need to show that, for the good restrictions, the event the "tail" variables x_T change the value of the function f_ρ, i.e. $\mathrm{sgn}(p_\rho(x_T) + p'(\rho)) \neq \mathrm{sgn}(p'(\rho))$ has probability at most β. This event has probability at most

$$\mathbf{Pr}_{x_T}[|p_\rho(x_T) - p'(\rho)| \geq \theta].$$

This is done in [15] using a concentration bound on the "tail", assuming full independence. Thus, in this case, we need to modify the argument since the projection of $\mathcal{D}$ on the "tail" variables is not uniform. However, a careful inspection of the parameters reveals that the concentration bound needed above actually holds even under an assumption of $O(d \log(1/\beta))$-independence for the "tail" x_T. In particular, given the upper bound on $\|p_\rho - p'_\rho\|_2$ and the lower bound on θ, it suffices to apply Theorem B.4 for $t = \log(1/\beta)^{d/2}$, which only requires $(dt^{2/d})$-wise independence. Hence, we are done in this case too.

The proof of the third statement remains essentially unchanged for the following reason: One proceeds by considering a random restriction of the variables of p up to the τ-critical index – which in this case is small. Hence, the distribution induced by $\mathcal{D}$ on this space is still uniform. Since the randomness is over these "head" variables, all the arguments remain intact and the claim follows. ■

F Appendix to Section 8

We show a generalization of Theorem 8.1 to the intersection of $m > 1$ halfspaces, which implies Theorem 8.1 as the special case $m = 2$.

Theorem 8.1 (restatement). *Let $m > 1$ be an integer. Let $H_i = \{x : \langle a_i, x\rangle > \theta_i\}$ for $i \in [m]$, with $\|a_i\|_2 = 1$ for all i. Let X be a vector of n i.i.d. Gaussians, and Y be a vector of k-wise independent Gaussians. Then for $k = \Omega(m^6/\varepsilon^2)$,*

$$|\mathbf{Pr}[X \in \cap_{i=1}^m H_i] - \mathbf{Pr}[Y \in \cap_{i=1}^m H_i]| < \varepsilon$$

Proof. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be the map $F(x) = (\langle a_1, x\rangle, \ldots, \langle a_m, x\rangle)$, and let R be the region $\{x : \forall i \; x_i > \theta_i\}$. Similarly as in the proof of Theorem 6.1, we simply show a chain of inequalities after setting $\rho = \varepsilon/m$ and $c = m/\rho$:

$$\mathbf{E}[I_R(F(X))] \approx_\varepsilon \mathbf{E}[\tilde{I}_R^c(F(X))] \approx_\varepsilon \mathbf{E}[\tilde{I}_R^c(F(Y))] \approx_\varepsilon \mathbf{E}[I_R(F(Y))]. \tag{F.1}$$

Note the maximum influence τ does not play a role since under the Gaussian measure we never need invoke the Invariance Principle. For the first inequality, observe $d_2(x, \partial R) \geq \min_i\{|x_i - \theta_i|\}$. Then by a union bound,

$$\mathbf{Pr}[d_2(F(X), \partial R) \leq w] \leq \mathbf{Pr}[\min_i\{|\langle a_i, X\rangle - \theta_i|\} \leq w] \leq \sum_{i=1}^m \mathbf{Pr}[|\langle a_i, X\rangle - \theta_i| \leq w],$$

which is $O(mw)$ by Theorem B.2 with $d = 1$. Now,

$$
\begin{aligned}
|\mathbf{E}[I_R(F(X))] - \mathbf{E}[\tilde{I}_R^c(F(X))]| &\leq \mathbf{E}[|I_R(F(X))] - \tilde{I}_R^c(F(X))|] \\
&\leq \mathbf{Pr}[d_2(F(X), \partial R) \leq 2\rho] \\
&\qquad + O\left(\sum_{s=1}^{\infty}\left(\frac{m^2}{c^2 2^{2s}\rho^2}\right)\cdot \mathbf{Pr}[d_2(F(X),\partial R) \leq 2^{s+1}\rho]\right) \qquad \text{(F.2)} \\
&= \mathbf{Pr}[d_2(F(X), \partial R) \leq 2\rho] + O\left(\sum_{s=1}^{\infty} 2^{-2s}\cdot \mathbf{Pr}[d_2(F(X),\partial R) \leq 2^{s+1}\rho]\right) \\
&= O(m\rho) \\
&= O(\varepsilon)
\end{aligned}
$$

where Eq. (F.2) follows from Theorem 4.10.

The last inequality in Eq. (F.1) is argued identically, except that we need to have anticoncentration of the $|\langle a_i, Y\rangle|$ in intervals of size no smaller than $\rho = \varepsilon/m$; this was already shown to hold under $O(1/\rho^p)$-wise independence in [25, Lemma 2.5] for any p-stable distribution, and the Gaussian is p-stable for $p = 2$.

For the middle inequality we use Taylor's theorem, as was done in Lemma 6.3. If we truncate the Taylor polynomial at degree-$(k-1)$ for k even, then by our derivative bounds on mixed partials of $\tilde{I}_R^c$ from Theorem 4.8, the error term is bounded by

$$
(2c)^k \cdot m^k \cdot \frac{\sum_{i=1}^m \mathbf{E}[\langle a_i, X\rangle^k]}{k!} \leq (cm)^k \cdot 2^{O(k)}/k^{k/2},
$$

with the inequality holding by Lemma 5.2, and the m^k arising as the analogue of the 4^k term that arose in Eq. (6.1). This is at most ε for k a sufficiently large constant times $(cm)^2$, and thus overall $k = \Omega(m^6/\varepsilon^2)$-wise independence suffices. ■

Remark F.1. Several improvements are possible to reduce the dependence on m in Theorem 8.1. We presented the simplest proof we are aware of which obtains a polynomial dependence on m, for clarity of exposition. See Section G.2 for an improvement on the dependence on m to quartic.

Our approach can also show that bounded independence fools the intersection of any constant number m of degree-2 threshold functions. Suppose the degree-2 polynomials are $p_1, \ldots, p_m$. Exactly as in Section 6 we decompose each p_i into $p_{i,1} - p_{i,2} + p_{i,3} + p_{i,4} + C_i$. We then define a region $R \subset \mathbb{R}^{4m}$ by $\{x : \forall i \in [m]\ x_{4i-3}^2 - x_{4i-2}^2 + x_{4i-1} + x_{4i} + C_i + \mathrm{tr}(A_{p_{i,3}}) > 0\}$, and the map $F : \mathbb{R}^n \to \mathbb{R}^{4m}$ by

$$
F(x) = (M_{p_1}(X), \ldots, M_{p_n}(X))
$$

for the map $M_p : \mathbb{R}^n \to \mathbb{R}^4$ defined in Section 6. The goal is then to show $\mathbf{E}[I_R(F(X))] \approx_\varepsilon \mathbf{E}[I_R(F(Y))]$, which is done identically as in the proof of Theorem 6.1. We simply state the theorem here:

Theorem F.2. Let $m > 1$ be an integer. Let $H_i = \{x : p_i(x) \geq 0\}$ for $i \in [m]$, for some degree-2 polynomials $p_i : \mathbb{R}^n \to \mathbb{R}$. Let X be a vector of n i.i.d. Gaussians, and Y be a vector of k-wise independent Gaussians with $k = \Omega(\mathrm{poly}(m)/\varepsilon^8)$. Then,

$$
|\mathbf{Pr}[X \in \cap_{i=1}^m H_i] - \mathbf{Pr}[Y \in \cap_{i=1}^m H_i]| < \varepsilon
$$

Identical conclusions also hold for X, Y being drawn from $\{-1,1\}^n$, since we can apply the decision tree argument from Theorem E.1 to each of the m polynomial threshold functions separately so that, by a union bound, with probability at least $1-m\tau'$ each of the m PTF restrictions is either τ'-close to a constant function, or is τ'-regular. Thus for whatever setting of τ sufficed for the case $m=1$ ($\tau=\varepsilon^2$ for halfspaces [14] and $\tau=\varepsilon^9$ for degree-2 threshold functions (Theorem 6.1)), we set $\tau'=\tau/m$ then argue identically as before.

G Various Quantitative Improvements

In the main body of the paper, at various points we sacrificed proving sharper bounds in exchange for clarity of exposition. Here we discuss various quantitative improvements that can be made in our arguments.

G.1 Improved FT-mollification In Theorem 4.8, we showed that for $F:\mathbb{R}^d\to\mathbb{R}$ bounded and $c>0$ arbitrary, $\|\partial^\beta \tilde{F}^c\|_\infty \le \|F\|_\infty \cdot (2c)^{|\beta|}$ for all $\beta\in\mathbb{N}^d$. We here describe an improvement to this bound. The improvement comes by sharpening our bound on $\|\partial^\beta B\|_1$.

We use the following fact, whose proof can be found in [16].

Fact G.1. For any multi-index $\alpha\in\mathbb{N}^d$,

$$\int_{\|x\|_2\le 1} x^\alpha dx = \begin{cases} 0 & \text{if some } \alpha_i \text{ is odd} \\ \frac{2\prod_{i=1}^d \Gamma\left(\frac{\alpha_i+1}{2}\right)}{(|\alpha|+d)\cdot\Gamma\left(\frac{|\alpha|+d}{2}\right)} & \text{otherwise} \end{cases}.$$

The following lemma is used in our sharpening of the upper bound on $\|\partial^\beta B\|_1$.

Lemma G.2. For a multi-index $\alpha\in\mathbb{N}^d$,

$$\|x^\alpha\cdot b\|_2 \le \sqrt{\frac{\alpha!\cdot 2^{O(|\alpha|+d)}}{(|\alpha|+d)^{|\alpha|}}}$$

Proof. By Fact G.1,

$$\begin{aligned}
\|x^\alpha\cdot b\|_2^2 &= C_d\cdot\int_{\|x\|_2\le 1}\left(x^{2\alpha} - 2\sum_{i=1}^d x_i^2x^{2\alpha} + 2\sum_{i<j}x_i^2x_j^2x^{2\alpha} + \sum_i x_i^4x^{2\alpha}\right)dx \\
&= \frac{2C_d}{|\alpha|+d}\cdot\Bigg[\frac{\prod_{i=1}^d\Gamma\left(\alpha_i+\frac12\right)}{\Gamma\left(|\alpha|+\frac d2\right)} - 2\frac{\sum_{i=1}^d\left(\prod_{j\ne i}\Gamma\left(\alpha_j+\frac12\right)\right)\Gamma\left(\alpha_i+\frac32\right)}{\Gamma\left(|\alpha|+\frac d2+\frac12\right)} \\
&\qquad + 2\frac{\sum_{i<j}\left(\prod_{\substack{k\ne i\\k\ne j}}\Gamma\left(\alpha_k+\frac12\right)\right)\Gamma\left(\alpha_i+\frac32\right)\Gamma\left(\alpha_j+\frac32\right)}{\Gamma\left(|\alpha|+\frac d2+\frac32\right)} \\
&\qquad + \frac{\sum_{i=1}^d\left(\prod_{j\ne i}\Gamma\left(\alpha_j+\frac12\right)\right)\Gamma\left(\alpha_i+\frac52\right)}{\Gamma\left(|\alpha|+\frac d2+\frac32\right)}\Bigg].
\end{aligned}$$

Write the above expression as

$$\frac{2C_d}{|\alpha|+d}\cdot[W(\alpha)-X(\alpha)+Y(\alpha)+Z(\alpha)]. \tag{G.1}$$

For $\alpha = 0$ we have

$$W(0) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2})}, \quad X(0) = d \cdot \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+\frac{1}{2})}, \quad Y(0) = d(d-1) \cdot \frac{\pi^{d/2}}{4 \cdot \Gamma(\frac{d}{2}+\frac{3}{2})}, \quad Z(0) = d \cdot \frac{3\pi^{d/2}}{4 \cdot \Gamma(\frac{d}{2}+\frac{3}{2})}.$$

Using the fact that $\Gamma(z+1) = z\Gamma(z)$, we can rewrite these as

$$\frac{W(0)}{\pi^{d/2}} = \frac{1}{\Gamma(\frac{d}{2})}, \quad \frac{X(0)}{\pi^{-d/2}} = \frac{d}{\Gamma(\frac{d}{2}+\frac{1}{2})}, \quad \frac{Y(0)}{\pi^{-d/2}} = \frac{d(d-1)}{2(d+1)\cdot\Gamma(\frac{d}{2}+\frac{1}{2})}, \quad \frac{Z(0)}{\pi^{-d/2}} = \frac{3d}{2(d+1)\cdot\Gamma(\frac{d}{2}+\frac{1}{2})}.$$

We thus have $W(0)-X(0)+Y(0)+Z(0) = \Omega(W(0)+Y(0)+Z(0))$. Since $2C_d(W(0)-X(0)+Y(0)+Z(0))/d = \|b\|_2^2 = 1$, it thus suffices to show that $(W(\alpha)+Y(\alpha)+Z(\alpha))/(W(0)+Y(0)+Z(0)) \le (\alpha! \cdot 2^{O(|\alpha|+d)}) \cdot (|\alpha|+d)^{-|\alpha|}$ for general α. This can be seen just by showing the desired inequality for $W(\alpha)/W(0)$, $Y(\alpha)/Y(0)$, and $Z(\alpha)/Z(0)$ separately. We do the calculation for $W(\alpha)/W(0)$ here; the others are similar.

We have

$$W(0) \ge \frac{2^{-O(d)}}{d^{d/2}}, \qquad W(\alpha) \le \frac{\alpha! \cdot 2^{O(|\alpha|+d)}}{(|\alpha|+d)^{|\alpha|+d/2}},$$

and thus

$$\frac{W(\alpha)}{W(0)} \le \frac{\alpha! \cdot d^{d/2} \cdot 2^{O(|\alpha|+d)}}{(|\alpha|+d)^{|\alpha|+d/2}} \le \frac{\alpha! \cdot 2^{O(|\alpha|+d)}}{(|\alpha|+d)^{|\alpha|}}.$$

■

Lemma G.3. For any $\beta \in \mathbb{N}^d$ with $|\beta| = \Omega(d)$, $\|\partial^\beta B\|_1 \le 2^{O(|\beta|)} \cdot \sqrt{\beta! \cdot |\beta|^{-|\beta|}}$.

Proof. The proof is nearly identical to the proof of Lemma 4.5. The difference is in our bound of $\|x^\alpha \cdot b\|_2$. In the proof of Lemma 4.5, we just used that $\|x^\alpha \cdot b\|_2 \le \|b\|_2 = 1$. However ,by Lemma G.2, we can obtain the sharper bound

$$\|x^\alpha \cdot b\|_2 \le 2^{O(|\alpha|+d)}\sqrt{\alpha! \cdot (|\alpha|+d)^{-(|\alpha|+d)}}.$$

We then have

$$\begin{aligned}\|x^\alpha \cdot b\|_2 \cdot \|x^{\beta-\alpha} \cdot b\|_2 &\le 2^{O(|\beta|)}\sqrt{\alpha! \cdot (|\alpha|+d)^{-(|\alpha|+d)} \cdot (\beta-\alpha)! \cdot (|\beta-\alpha|+d)^{-(|\beta-\alpha|+d)}} \\ &\le 2^{O(|\beta|)}\sqrt{\beta! \cdot |\beta|^{-|\beta|}}\end{aligned}$$

■

We now have the following sharpening of item (i) from Theorem 4.8. Over high dimension, for some β the improvement can be as large as a shrinking of our upper bound in Theorem 4.8 by a $d^{-|\beta|/2}$ factor (for example, when each β_i is $|\beta|/d$).

Theorem G.4. Let $F : \mathbb{R}^d \to \mathbb{R}$ be bounded and $c > 0$ be arbitrary, and $\beta \in \mathbb{N}^d$ have $|\beta| = \Omega(d)$. Then,

$$\|\partial^\beta \tilde{F}^c\|_\infty \le \|F\|_\infty \cdot c^{|\beta|} \cdot 2^{O(|\beta|)} \cdot \sqrt{\beta! \cdot |\beta|^{-|\beta|}}$$

Proof. Note in Eq. (4.6) in the proof of Theorem 4.8, we showed that $\|\partial^\beta \tilde{F}^c\|_\infty \le \|F\|_\infty \cdot c^{|\beta|} \cdot \|\partial^\beta B\|_1$. The claim then follows by applying Lemma G.3 to bound $\|\partial^\beta B\|_1$. ■

G.2 Improvements to fooling the intersection of halfspaces

In the proof of Theorem 8.1 in Section F, we presented a proof showing that $\Omega(m^6/\varepsilon^2)$-independence ε-fools the intersection of m halfspaces under the Gaussian measure. In fact, this dependence on m can be improved to quartic. One factor of m is shaved by using the improved bound from Theorem G.4, and another factor of m is shaved by a suitable change of basis. The argument used to shave the second factor of m is specific to the Gaussian case, and does not carry over to the Bernoulli setting.

Theorem G.5. Let $m > 1$ be an integer. Let $H_i = \{x : \langle a_i, x\rangle > \theta_i\}$ for $i \in [m]$, with $\|a_i\|_2 = 1$ for all i. Let X be a vector of n independent standard normals, and Y be a vector of k-wise independent Gaussians. Then for $k = \Omega(m^4/\varepsilon^2)$ and even,

$$|\mathbf{Pr}[X \in \cap_{i=1}^m H_i] - \mathbf{Pr}[Y \in \cap_{i=1}^m H_i]| < \varepsilon$$

Proof. Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be an orthonormal basis for a linear space containing the a_i. Define the region $R = \{x : \forall i \in [m]\ \sum_{j=1}^m \langle a_i, v_j\rangle x_j > \theta_i\}$ in $\mathbb{R}^m$. Note R is itself the intersection of m halfspaces in $\mathbb{R}^m$, with the ith halfspace having normal vector $b_i \in \mathbb{R}^m$ with $(b_i)_j = \langle a_i, v_j\rangle$.

We now define the map $F : \mathbb{R}^n \to \mathbb{R}^m$ by $F(x) = (\langle x, v_1\rangle, \ldots, \langle x, v_m\rangle)$. It thus suffices to show that $\mathbf{E}[I_R(F(X))] \approx_\varepsilon \mathbf{E}[I_R(F(Y))]$. We do this by a chain of inequalities, similarly as in the proof of Theorem 8.1. Below we set $c = m^2/\varepsilon$.

$$\mathbf{E}[I_R(F(X))] \approx_\varepsilon \mathbf{E}[\tilde{I}_R^c(F(X))] \approx_\varepsilon \mathbf{E}[\tilde{I}_R^c(F(Y))] \approx_\varepsilon \mathbf{E}[I_R(F(Y))]. \tag{G.2}$$

For the first inequality and last inequalities, since we performed an orthonormal change of basis the $F(X)_i$ remain independent standard normals, and we can reuse the same analysis from the proof of Theorem 8.1 without modification.

For the middle inequality we use Taylor's theorem. If we set $R(F(x)) = |\tilde{I}_R^c(F(x)) - P_{k-1}(F(x))|$ for P_{k-1} the degree-$(k-1)$ Taylor polynomial approximating $\tilde{I}_R^c$, then

$$\begin{aligned} R(x) &\le \sum_{|\beta|=k} \|\partial^\beta \tilde{I}_R^c\|_\infty \cdot \frac{\prod_{i=1}^m |x_i|^{\beta_i}}{\beta!} \\ &\le \frac{2^{O(k)} \cdot c^k}{k^{k/2}} \cdot \sum_{|\beta|=k} \frac{\prod_{i=1}^m |x_i|^{\beta_i}}{\sqrt{\beta!}} \end{aligned} \tag{G.3}$$

by Theorem G.4. Now note

$$\begin{aligned} \sum_{|\beta|=k} \frac{\prod_{i=1}^m |x_i|^{\beta_i}}{\sqrt{\beta!}} &= \frac{1}{k!} \cdot \sum_{|\beta|=k} \sqrt{\beta!} \cdot \binom{k}{\beta} \cdot |x|^\beta \\ &\le \frac{2^{O(k)} \cdot k^m}{k^k} \cdot \sum_{\substack{|\beta|=k \\ \forall i\ 2|\beta_i}} \sqrt{\beta!} \cdot \binom{k}{\beta} \cdot |x|^\beta \qquad \text{(G.4)} \\ &= \frac{2^{O(k)}}{k^k} \cdot \sum_{|\beta|=k/2} \sqrt{(2\beta)!} \cdot \binom{k}{2\beta} \cdot |x|^{2\beta} \\ &\le \frac{2^{O(k)}}{k^{k/2}} \cdot \sum_{|\beta|=k/2} \binom{k/2}{\beta} \cdot |x|^{2\beta} \end{aligned}$$

$$= \frac{2^{O(k)}}{k^{k/2}} \cdot \|x\|_2^k \tag{G.5}$$

where $|x|$ denotes the vector $(|x|_1, \ldots, |x|_m)$. Eq. (G.4) holds for the following reason. Let $\beta \in \mathbb{N}^m$ be arbitrary. Since k is even, the number of odd β_i must be even. Let M be any perfect matching of the indices i with odd β_i. Then for $(i,j) \in M$, either $|x_i|^{\beta_i+1}|x_j|^{\beta_j-1}$ or $|x_i|^{\beta_i-1}|x_j|^{\beta_j+1}$ must be at least as large as $|x_i|^{\beta_i}|x_j|^{\beta_j}$. Let β' be the new multi-index with only even indices obtained by making all such replacements for $(i,j) \in M$. We then replace $\sqrt{\beta!} \cdot \binom{k}{\beta} \cdot |x|^\beta$ in the summation with $\sqrt{\beta'!} \cdot \binom{k}{\beta'} \cdot |x|^{\beta'}$. In doing so, we have $x^{\beta'} \geq x^\beta$, but $\sqrt{\beta'!} \cdot \binom{k}{\beta'}$ may have decreased from $\sqrt{\beta!} \cdot \binom{k}{\beta}$, but by at most a $2^{O(k)}k^m$ factor since each β_i decreased by at most 1 and is at most k. Also, in making all such replacements over all $\beta \in \mathbb{N}^m$, we must now count each β with even coordinates at most 3^m times, since no such β can be mapped to by more than 3^m other multi-indices (if we replaced some multi-index with β, that multi-index must have its ith coordinate either one larger, one smaller, or exactly equal to β_i for each i). Note subsequent inequalities dropped the k^m term in the numerator since $2^{O(k)} \cdot k^m = 2^{O(k)}$ for our choice of k.

Now by Eq. (G.5),

$$\mathbf{E}\left[\sum_{|\beta|=k} \frac{\prod_{i=1}^n |F(X)_i|^{\beta_i}}{\sqrt{\beta!}}\right] \leq \frac{2^{O(k)}}{k^{k/2}} \cdot \mathbf{E}\left[\left(\sum_{i=1}^m F(X)_i^2\right)^{k/2}\right]$$

Since the $F(X)_i$ are independent standard normal random variables, $\sum_{i=1}^m F(X)_i^2$ follows a chi-squared distribution with m degrees of freedom, and its $k/2$th moment is determined by k-wise independence, and thus

$$\mathbf{E}\left[\left(\sum_{i=1}^m F(X)_i^2\right)^{k/2}\right] = 2^{k/2} \cdot \frac{\Gamma(k/2 + m/2)}{\Gamma(m/2)} = 2^{O(k)} \cdot k^m \cdot k^{k/2} \leq 2^{O(k)} \cdot k^{k/2}. \tag{G.6}$$

This finishes our proof, since by Eq. (G.3) the expected value of our Taylor error is

$$\frac{2^{O(k)} \cdot c^k}{k^{k/2}} \cdot \mathbf{E}\left[\sum_{|\beta|=k} \frac{\prod_{i=1}^m |F(X)_i|^{\beta_i}}{\sqrt{\beta!}}\right] = \frac{2^{O(k)} \cdot c^k}{k^{k/2}} \cdot \left(\frac{2^{O(k)}}{k^{k/2}} \cdot 2^{O(k)} \cdot k^{k/2}\right) = \frac{2^{O(k)} \cdot c^k}{k^{k/2}},$$

which is $O(\varepsilon)$ for $k = \Omega(c^2) = \Omega(m^4/\varepsilon^2)$. ■