EECS 225A Spring 2005

Homework 6 solutions

1. Hayes problem 4.7

Solution

(a) Note that $E(z) = Y(z)A(z) - X(z)B(z)$, so

$$c(n) = \sum_{k=0}^{\infty} a(k)y(n-k) - \sum_{k=0}^{\infty} b(k)x(n-k)$$

With

$$E = \sum_{n=0}^{\infty} c^2(n)$$

the Normal Equations are found by setting the derivatives of $E$ with respect to $a(k)$ and $b(k)$ equal to zero,

$$\frac{\partial E}{\partial a(k)} = 0 \quad ; \quad \frac{\partial E}{\partial b(k)} = 0$$

Thus,

$$\frac{\partial E}{\partial a(k)} = \sum_{n=0}^{\infty} 2c(n)y(n-k) = 2 \sum_{n=0}^{\infty} \left\{ \sum_{l=0}^{p} a(l)y(n-l) - \sum_{l=0}^{q} b(l)x(n-l) \right\} y(n-k) = 0$$

and

$$\frac{\partial E}{\partial b(k)} = -\sum_{n=0}^{\infty} 2c(n)x(n-k) = -2 \sum_{n=0}^{\infty} \left\{ \sum_{l=0}^{p} a(l)y(n-l) - \sum_{l=0}^{q} b(l)x(n-l) \right\} x(n-k) = 0$$
Dividing by two, and rearranging the sums, we have
\[
\sum_{l=0}^{p} a(l) \left( \sum_{n=0}^{\infty} y(n-l)y(n-k) \right) - \sum_{l=0}^{q} b(l) \left( \sum_{n=0}^{\infty} x(n-l)y(n-k) \right) = 0 \quad ; \quad k = 1, \ldots, p
\]
and
\[
-\sum_{l=0}^{p} a(l) \left( \sum_{n=0}^{\infty} y(n-l)x(n-k) \right) + \sum_{l=0}^{q} b(l) \left( \sum_{n=0}^{\infty} x(n-l)x(n-k) \right) = 0 \quad ; \quad k = 0, \ldots, q
\]
If we define
\[
\tau_y(k, l) = \sum_{n=0}^{\infty} x(n-l)y(n-k)
\]
\[
\tau_y(k, l) = \sum_{n=0}^{\infty} y(n-l)y(n-k)
\]
\[
\tau_x(k, l) = \sum_{n=0}^{\infty} x(n-l)x(n-k)
\]
then these equations become
\[
\sum_{l=0}^{p} a(l) \tau_y(k, l) - \sum_{l=0}^{q} b(l) \tau_y(k, l) = 0 \quad ; \quad k = 1, 2, \ldots, p
\]
\[
-\sum_{l=0}^{p} a(l) \tau_x(k, l) + \sum_{l=0}^{q} b(l) \tau_x(k, l) = 0 \quad ; \quad k = 0, 1, \ldots, q
\]
Assuming that the coefficients have been normalized so that \(a(0) = 1\), we have
\[
\sum_{l=0}^{p} a(l) \tau_y(k, l) - \sum_{l=0}^{q} b(l) \tau_y(k, l) = -\tau_x(k, 0) \quad ; \quad k = 1, 2, \ldots, p
\]
\[
-\sum_{l=0}^{p} a(l) \tau_x(k, l) + \sum_{l=0}^{q} b(l) \tau_x(k, l) = \tau_y(k, 0) \quad ; \quad k = 0, 1, \ldots, q
\]
Writing these in matrix form we obtain
\[
\begin{bmatrix}
R_p & -R_y \\
-R_y & R_x
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix} =
\begin{bmatrix}
-R_x \\
\tau_y
\end{bmatrix}
\]
where \(a^T = [a(1), a(2), \ldots, a(p)]\), \(b^T = [b(0), b(1), \ldots, b(q)]\), \(\tau_x^T = [\tau_x(1, 0), \tau_x(2, 0), \ldots, \tau_x(p, 0)]\), and \(\tau_y^T = [\tau_y(1, 0), \tau_y(2, 0), \ldots, \tau_y(q, 0)]\). Also, \(R_x\) is a \(p \times p\) matrix with entries \(r_x(k, l)\), \(R_y\) is a \((q+1) \times (q+1)\) matrix with entries \(r_y(k, l)\), and \(R_{xy}\) is a \(p \times (q+1)\) matrix with entries \(r_{xy}(k, l)\).

(b) Suppose \(S(z) = C(z)/D(z)\). Then
\[
B(z) = B(z)X(z) - \frac{C(z)}{D(z)} A(z)X(z)
\]
and the error can be made equal to zero if
\[
\frac{B(z)}{A(z)} = \frac{C(z)}{D(z)}
\]

2. Hayes problem 4.10

Solution
The equations for the coefficients \( a_p(k) \), \( k = 1, \ldots, p \), that minimize the error \( \mathcal{E}_p \) are found by setting the derivatives of \( \mathcal{E}_p \) with respect to \( a_p(k) \) equal to zero. Thus, assuming that \( x(n) \) is real, we have

\[
\frac{\partial \mathcal{E}_p}{\partial a_p(k)} = \sum_{n=0}^{\infty} 2c(n)x(n - k - N) = 0
\]

Dividing by two, and substituting for \( c(n) \), we have

\[
\sum_{n=0}^{\infty} \left[ x(n) + \sum_{l=1}^{p} a_p(l)x(n - l - N) \right] x(n - k - N) = 0
\]

or

\[
\sum_{l=1}^{p} a_p(l) \left[ \sum_{n=0}^{\infty} x(n - l - N)x(n - k - N) \right] = -\sum_{n=0}^{\infty} x(n)x(n - k - N)
\]

If we define

\[
r_p(k, l) = \sum_{n=0}^{\infty} x(n - l)x(n - k)
\]

then it is easily shown that \( r_p(k, l) \) depends only on the difference, \( k - l \), and we may write

\[
r_p(k) = \sum_{n=0}^{\infty} x(n)x(n - k)
\]

Thus, the normal equations become

\[
\sum_{l=1}^{p} a_p(l)r_p(k - l) = -r_p(k + N)
\]

Finally, using the orthogonality condition

\[
\sum_{n=0}^{\infty} c(n)x(n - k - N) = 0
\]

we have, for the minimum error,

\[
\{ \mathcal{E}_p \}_{\text{min}} = \sum_{n=0}^{\infty} c(n) \left[ x(n) + \sum_{l=1}^{p} a_p(l)x(n - l - N) \right] = \sum_{n=0}^{\infty} c(n)x(n)
\]

Therefore,

\[
\{ \mathcal{E}_p \}_{\text{min}} = \sum_{n=0}^{\infty} x(n) + \sum_{l=1}^{p} \sum_{n=0}^{\infty} a_p(l)x(n - l - N) x(n) = r_p(0) + \sum_{l=1}^{p} a_p(l)r_p(l + N)
\]

3. Hayes problem 4.21
If we define $a_p(0) = 1$, then the error $e(n)$ is 

$$
e(n) = a_p(n) \cdot x(n) - b(0) \cdot y_{00}(n) = \sum_{l=0}^{p} a_p(l) x(n-l) - b(0) \left[ \delta(n) + \delta(n - n_0) \right]$$

and the mean-square error that we want to minimize is 

$$\xi_p = \sum_{n=0}^{2n_0-1} e^2(n) = \sum_{n=0}^{2n_0-1} \left[ \sum_{l=0}^{p} a_p(l) x(n-l) - b(0) \delta(n) - b(0) \delta(n-n_0) \right]^2$$

Setting the derivative with respect to $a_p(k)$ equal to zero, we have 

$$\frac{\partial E}{\partial a_p(k)} = \sum_{n=0}^{2n_0-1} 2 \left[ \sum_{l=0}^{p} a_p(l) x(n-l) - b(0) \delta(n) - b(0) \delta(n-n_0) \right] x(n-k) = 0$$

If we define 

$$r_x(k, l) = \sum_{n=0}^{2n_0-1} x(n-l) x(n-k)$$

then the normal equations become (recall that $a_p(0) = 1$) 

$$\sum_{l=1}^{p} a_p(l) r_x(k, l) - b(0) x(-k) - b(0) x(n_0 - k) = -r_x(k, 0) \quad ; \quad k = 1, 2, \ldots, p$$

Assuming that $x(n) = 0$ for $n < 0$, with $x = [x(n_0 - 1), x(n_0 - 2), \ldots, x(n_0 - p)]^T$, the normal equations may be written in matrix form as follows 

$$R_x a - b(0) x = -r_x$$

Finally, differentiating with respect to $b(0)$ we have 

$$\frac{\partial E}{\partial b(0)} = -\sum_{n=0}^{\infty} 2 \left[ \sum_{l=0}^{p} a_p(l) x(n-l) - b(0) \delta(n) - b(0) \delta(n-n_0) \right] \delta(n) + \delta(n-n_0)]$$

Thus, 

$$x(0) - b(0) + \sum_{l=1}^{p} a_p(l) x(n_0 - l) - b(0) = -x(n_0)$$

or, in vector form, we have 

$$x^T a - 2b(0) = -x(0) - x(n_0)$$

Putting all of these together in matrix form yields 

$$\begin{bmatrix} R_x & x \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} = -\begin{bmatrix} r_x \\ x^T 1 \end{bmatrix}$$

4. Hayes problem 4.25
(a) As we did in Example 4.7.1, we would like to find an ARMA(1,1) model for \( x(n) \) that has the given autocorrelation values. Since the Yule-Walker equations are

\[
\begin{bmatrix}
    r_x(0) & r_x(1) \\
    r_x(1) & r_x(0)
\end{bmatrix}
\begin{bmatrix}
    1 \\
    a_1(1)
\end{bmatrix}
= \begin{bmatrix}
    c_1(0) \\
    c_1(1)
\end{bmatrix}
\]

then the modified Yule-Walker equations for \( a(1) \) are

\[ r_x(1)a(1) = -r_x(2) \]

which gives \( a_1(1) = -r_x(2)/r_x(1) = -1/2 \).

For the moving average coefficients, we begin by computing \( c_1(0) \) and \( c_1(1) \) using the Yule-Walker equations as follows

\[
\begin{bmatrix}
    r_x(0) & r_x(1) \\
    r_x(1) & r_x(0)
\end{bmatrix}
\begin{bmatrix}
    1 \\
    a_1(1)
\end{bmatrix}
= \begin{bmatrix}
    c_1(0) \\
    c_1(1)
\end{bmatrix}
\]

With the given values for \( r_x(k) \), using \( a_1(1) = -1/2 \), we find

\[
\begin{bmatrix}
    c_1(0) \\
    c_1(1)
\end{bmatrix}
= \begin{bmatrix}
    3 & \frac{9}{4} \\
    \frac{9}{4} & 3
\end{bmatrix}
\begin{bmatrix}
    1 \\
    -1/2
\end{bmatrix}
= \begin{bmatrix}
    15/8 \\
    3/4
\end{bmatrix}
\]

and

\[ [C_1(z)]_+ = \frac{3}{8} + \frac{3}{4}z^{-1} \]

Multiplying by \( A_1^*(1/z^*) = 1 - \frac{1}{2}z \) we have

\[ [C_1(z)]_+ A_1^*(1/z^*) = (1 - \frac{15}{8} + \frac{9}{8}z^{-1}) (1 - \frac{1}{2}z) = -\frac{15}{16}z + \frac{9}{8} + \frac{3}{4}z^{-1} \]

Therefore, the causal part of \( P_x(z) \) is

\[ [P_x(z)]_+ = \left[ [C_1(z)]_+ A_1^*(1/z^*) \right]_+ = \frac{3}{8} + \frac{3}{4}z^{-1} \]

Using the symmetry of \( P_x(z) \), we have

\[ C_1(z)A_1^*(1/z^*) = B(z)B^*(1/z^*) = \frac{3}{4}z + \frac{9}{8} + \frac{3}{4}z^{-1} \]

Performing a spectral factorization gives

\[ P_x(z) = B(z)B^*(1/z^*) = \frac{3}{4}(1 + z^{-1})(1 + z) \]

so the ARMA(1,1) model is

\[ H(z) = \frac{\sqrt{3}}{2} \frac{1 + z^{-1}}{1 - \frac{1}{2}z^{-1}} \]

(b) Yes. The model matches \( r_x(k) \) for \( k = 0, 1, 2 \), and for \( k > 2 \) note that

\[ r_x(k) = \frac{1}{2}r_x(k - 1) \]

which they do.