W4231: Analysis of Algorithms

9/14/1999

- Median Selection

Definition of median

Let $A = a_1 \cdots a_n$ be a sequence of integers.

The median of A is a value v such that

$$|\{i : a_i < v\}| \leq n/2$$

and

$$|\{i : a_i > v\}| \leq n/2$$

That is, if $b_1 \cdots b_n$ is A sorted in ascending order, then the median is $b_{\lfloor n/2 \rfloor}$.

Algorithmic problem

We want to compute the median using only comparisons.

More generally, given k we would like to find a value a such that

$$|\{i : a_i < v\}| \leq k$$

and

$$|\{i : a_i > v\}| \leq n - k$$

A $O(n \log n)$ solution

- Sort A and return the value in the $\lfloor n/2 \rfloor$-th (respectively, k-th) position.

This requires time $O(n \log n)$.

A procedure inspired by quicksort

Assume elements are *distinct* for the moment.

Select $(A[1], \ldots, A[n], k)$
begin
  $v \leftarrow$ ChoosePivot $(A[1], \ldots, A[n])$;
  $i \leftarrow$ Partition $(A[1], \ldots, A[n], v)$;
  if $i = k$ **then return** v
  else if $i > k$ **then** Select$(A[1], \ldots, A[i], k)$
  else Select$(A[i+1], \ldots, A[n], k - i)$
end

ChoosePivot() is a procedure that decides which value to partition around.

Partition() does the partition and returns the index where the pivot has been placed.

Remember Quicksort

```
QuickSort (A[1],...,A[n])
begin
  if n = 1 then halt;
  v ← ChoosePivot (A[1,...,n]);
  i ← Partition (A[1,...,n], v);
  QuickSort(A[1],...,A[i − 1]);
  QuickSort(A[i + 1],...,A[n])
end
```


Implementing Partition in $O(n)$ Time

```
Partition (A[1],...,A[n], v)
begin
  i ← 1; j ← n;
  while true do begin
    repeat (i ← i + 1) until A[i] ≥ v;
    repeat (j ← j − 1) until A[j] ≤ v;
    if (i < j) then swap A[i] and A[j]
    else return i
  end
end
```


Implementing ChoosePivot()

- Choose always the first element.

  There can be cases where the selection procedure takes $O(n^2)$ time. Similar problem with Quicksort. Like for Quicksort, the *average case* is better.

- Choose a random element in the array.

  Average time for Select is $O(n)$. Average time for QuickSort is $O(n \log n)$. Will do analysis next time.

- Choose an element that is guaranteed to be bigger than $\geq 30\%$ of the elements and smaller than $\geq 30\%$ of the elements.

  Worst case Select $O(n)$. Worst case QuickSort $O(n \log n)$. How to implement?

The median of medians

Divide the vector into $n/5$ subsequences of 5 consecutive elements each.

Find the median in each sequence. Let $m_1, \ldots, m_{n/5}$ be these medians. Find recursively the median of these medians, let it be mm. This will be the pivot.


```
ChoosePivotBFPRT (A[1],...,A[n])
begin
  for i = 1 to n/5 do
    let m_i be the median of A[5i − 4], A[5i − 3],...,A[5i];
  mm = Select(m_1,...,m_{n/5}, n/10);
  return mm
end
```

Analysis

Consider ChoosePivotBFPRT$(A[1], \ldots, A[n])$.

Call "intermediate medians" the values $m_1, \ldots, m_{n/5}$.

There are $n/10$ intermediate medians $\leq mm$. For each one, there are two elements smaller than them. Thus there are $.3n$ elements $< mm$

Likewise, there are $.3n$ elements $\geq mm$.

In Select, we use $T(n/5) + O(n)$ time to compute ChoosePivotBFPRT(), then $O(n)$ time for Partition() and then we recurse on a sub-instance of size at most $.7n$.

$T(n) \leq T(n/5) + T(.7n) + cn$.

This solves to $T(n) \leq 10cn$.

Taking into account $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$

In the general case, n may not be divisible by 5.

We have to solve $\lceil n/5 \rceil$ median subproblems (the last one may involve less than 5 elements), and then find the median of these intermediate medians, which takes time $T(\lceil n/5 \rceil)$.

The median-of-medians is bigger than at least

$3(\lceil 1/2 \lceil n/5 \rceil \rceil - 2) \geq .3n - 6$

elements in the array; and smaller than at least that many ones.

Running time is

$T(n) \leq T(\lceil n/5 \rceil) + T(.7n + 6) + O(n)$

that still solves to $T(n) = O(n)$.

If there are repeated elements

We can reduce to the case of no repetitions by considering the median of the array $a'_1 \cdots a'_n$, where $a'_i = (n+1)a_i + i$.

The order is preserved and there are no repetitions.

Alternatively, one has to refine the algorithm and the analysis (see CLR).

Why 5?

In general, the recursion

$$T(n) \leq T(\alpha n) + T(\beta n) + cn \ , \ T(1) = c'$$

solves to $T(n) = O(n)$ if $\alpha + \beta < 1$.

While a recursion

$$T(n) \leq T(\alpha n) + T(\beta n) + cn \ , \ T(1) = c'$$

with $\alpha + \beta \geq 1$ typically yields $T(n) = \Omega(n \log n)$.

3 does not work

Dividing the array in groups of 3 elements, we would spend $T(n/3)$ time in finding the median-of-medians.

Then, even if the size of the vector is a multiple of 3, we can only guarantee that the median-of-medians is larger than $n/3$ elements and smaller than $n/3$.

So we may recurse to a sub-array with $n - 2n/3$ elements. The recursion is

$T(n) \leq T(n/3) + T(2n/3) + O(n)$

No good!

Lower bounds

We need to make at least $n/2$ comparisons just to read all the elements.

More involved argument: we need $\geq n - 1$ comparisons.

Much more involved argument: we need $\geq 2n - o(n)$ comparisons. Bent and John (1985)

Exceedingly complicated: we need $\geq (2 + 2^{-30})n - o(n)$ comparisons, Dor and Zwick (1997).

Better (?) algorithms

The median-of-medians algorithm is by Blum, Floyd, Pratt, Rivest, Tarjan (1973).

An algorithm that makes $5n + o(n)$ comparisons is due Schönhage, Pippenger, Paterson (1976).

Same people, same year, an algorithm that makes $3n + o(n)$ comparisons, Schönhage, Pippenger, Paterson (1976).

Quite recently: an algorithm that makes $2.95n + o(n)$ comparisons, due to Dor and Zwick (1995).