Notes for Lecture 8

In the previous lectures, we have seen that in the reduction form MAX-3-SAT to MAX-3-SAT where each variable occurs a bounded number of times we required the construction of a graph $G(V, E)$ with the following properties:

- There is a constant d such that for every n there is a graph $G(V, E), |V| = n$ with much degree d such that $\forall S \subseteq V, |S| \leq \frac{|V|}{2}$ hte number of edges with one endpoint in S and one endpoint in $V - S$ is $\geq |S|$.

In addition, we required that $\forall n$ this graph should be efficiently constructed. Note that for our purposes, multigraphs are allowed.

Definition 1 *(Edge-expansion of a graph) We define the edge-expansion of a graph G :*

$$h(G) = min_{|S|\leq|V|/2} \frac{edges(S, V - S)}{|S|}$$

In what follows, we consider $G = (V, E)$ to be a given graph and $M \in \mathbb{R}^{V\times V}$ its adjacency matrix, that is

$$M(u, v) := \text{ number of edges between } u \text{ and } v \tag{1}$$

Note that M is symmetric.

Definition 2 *If $M \in \mathbb{C}^{n\times n}$, $\lambda \in \mathbb{C}, x \in \mathbb{C}^n$ and $xM = \lambda x$ then λ is an eigenvalue of M and x is an eigenvector of M.*

Example 1 *Let M be the adjacency matrix of a d-regular graph. Then $(1, 1, \cdots, 1) \cdot M = (d, d, \cdots, d) = d(1, 1, \cdots, 1)$. Therefore, the vector $(1, 1, \cdots, 1)$ is an eigenvector of M with corresponding eigenvalue 1.*

Generally, $xM = \lambda x \Rightarrow x(M - \lambda I) = 0 \Rightarrow det(M - \lambda I) = 0$. $det(M - \lambda I)$ is a polynomial in λ over $\mathbb{C}$ of degree n, and it has n roots (counting multiplicities). Therefore, λ is an eigenvalue of M iff it is a root of $det(M - \lambda I)$ and so, counting multiplicities, M has n eigenvalues.

Theorem 3 *If $M \in \mathbb{R}^{n\times n}$ is symmetric then the following properties hold:*

1. *all n eigenvalues $\lambda_1, \cdots, \lambda_n$ are real*
2. *one can find an orthogonal set of eigenvectors $x_1, \cdots, x_n$ such that x_i has corresponding eigenvalue λ_i and $x_i \perp x_j$ for $i \neq j$.*

We note that a multiple of an eigenvector is also an eigenvector and therefore we can assume w.l.o.g. that all the x_i have length one.

Lemma 4 *Let $M \in \mathbb{R}^{n\times n}$ symmetric. Then $\lambda_1 = max_{x\in\mathbb{R}^n,\|x\|=1}\{xMx^T\}$, where $xMx^T = \sum_{i,j} x(i)x(j)M(i,j)$*

PROOF:

- (a) Assume $\lambda_1 \geq \lambda_2 \cdots, \geq \lambda_n$. Then $x_1Mx_1^T = \lambda_1 x_1 x_1^T = \lambda_1$ therefore, $max_{x\in\mathbb{R}^n,||x||=1}\{xMx^T\} \geq \lambda_1$.
- (b) Conversely, let x be any vector of length one, $x \in \mathbb{R}^n$, $\|x\| = 1$. Let $x = a_1x_1 + a_2x_2 + \cdots + a_nx_n$.

$$xMx^T = \sum_{i,j} x(i)x(j)M(i,j) = (\sum_i a_ix_i)M(\sum_i a_ix_i)^T =$$

$$(\sum_i \lambda_i a_i x_i)(\sum_j a_i x_j)^T = \sum_i \lambda_i a_i^2 \leq max_i\lambda_i \sum_i a_i^2 = \lambda_1$$

Therefore $max_{x\in\mathbb{R}^n,||x||=1}\{xMx^T\} \leq \lambda_1$.

□

We can also prove that $\lambda_2 = max_{x\in\mathbb{R}^n,\|x\|=1,x\perp x_1}\{xMx^T\}$. For (a) use $x = x_2$, and conclude $max_{x\in\mathbb{R}^n,\|x\|=1,x\perp x_1}\{xMx^T\} \geq \lambda_2$.
For (b) take any $x \in \mathbb{R}^n, \|x\| = 1, x\perp x_1$.
Let $x = a_1x_1 + a_2x_2 + \cdots + a_nx_n$. Then $xMx^T = \sum_{i=2}^n \lambda_i a_i^2 = \lambda_2$.
A similar argument shows that

$$\max\{|\lambda_2|, \ldots, |\lambda_n|\} = \max_{x\perp x_1,\|x\|=1} |xMx^T| \tag{2}$$

Theorem 5 *Let G be a d-regular graph and M its adjacency matrix. Let $\lambda_1, \cdots, \lambda_n$ its eigenvalues and $x_1, \cdots, x_n$ the corresponding eigenvectors. Then $\lambda_1 = d$.*

PROOF: Trivially, $\lambda_1 \geq d$ because d is an eigenvalue for some i.
Let $x \in \mathbb{R}^n, ||x|| = 1, xM = \lambda_1 x$

$$0 \leq \sum_{u,v} M(u,v)(x(u) - x(v))^2 = 2d\sum_v x(v)^2 - 2\sum_{u,v} x(u)x(v)M(u,v)$$

$$= 2d\|x\|^2 - 2xMx^T = 2d - 2\lambda_1 \Rightarrow d \geq \lambda_1$$

Since $d \leq \lambda_1$ and $d \geq \lambda_1$ it follows $d = \lambda_1$. $\square$

It is helpful to think of the vector x as a labelling of the graph. So far, we have proved that the largest eigenvalue is d. Now we will prove that if the second largest eigenvalue is also equal to d then the graph is disconnected.
To see this fact, choose $x_1 = \frac{1}{\sqrt{n}}(1, 1 \cdots, 1)$ and x_2 another eigenvector orthogonal to x_1. x_2 should be $(x_2(1), \cdots, x_2(n))$ with $\sum_i x_2(i) = 0$. Therefore, some entries should be positive and some others should be negative(*).

$$0 \leq \sum_{u,v} M(u,v)(x_2(u) - x_2(v))^2 = 2d - 2\lambda_2 = 0$$

Therefore, for x_2 any two adjacent vertices must have identical labels and the only way for condition (*) to hold is the graph to be disconnected.
Conversely, if the graph is disconnected then $\lambda_1 = \lambda_2 = d$ (Exercise).

We now turn the discussion to the edge-expansion of the graph (definition 1). Observe that $h(G) = 0$ iff graph is disconnected, equivalently iff $\lambda_2 = d$. Assume that $\lambda_1 - \lambda_2 > \epsilon$. Then $h(G) > \epsilon'$. In fact,

Theorem 6 $\lambda_2 \geq d - 2h \Rightarrow h \geq \frac{d-\lambda_2}{2}$

PROOF: Let S be the set that achieves $h(G) = \frac{edges(S,V-S)}{|S|}$
Remember that $\lambda_2 = \lambda_2 = max_{x \in \mathbb{R}^n, \|x\|=1, x \perp x_1}\{xMx^T\}$
Define x' based on S, such that $x' \perp (1, 1, \cdots, 1)$.
Prove that $x'Mx'^T \geq (d - 2h) \cdot \|x'\|^2$.
For $x = \frac{x'}{\|x'\|^2}$ we have $xMx^T \geq d - 2h \Rightarrow \lambda_2 \geq d - 2h$. $\square$

Theorem 7 $h \leq \sqrt{d(d - \lambda_2)} \Rightarrow h^2 \geq d(d - \lambda_2)$.

Before we see the proof of the latest theorem let's consider the solution to our previous exercise.
Assume G is disconnected with S and $V - S$ the two connected components.
Let $p = \frac{|S|}{|V|}$, $q = \frac{|V-S|}{|V|}$. Assign

$$x(v) = \begin{cases} q & \text{if } v \in S \\ -p & \text{if } v \notin S \end{cases}$$

First, observe that $x \perp (1, 1, \cdots, 1)$ since $\sum_v x(v) = q \cdot |S| - p \cdot |V - S| = qpn - pqn = 0$.
Second, look at $xM = (\underbrace{dq, dq, \cdots, dq}_{|S|}, \underbrace{-pd, -pd, \cdots, -pd}_{|V-S|}) = dx$.

Therefore, if the graph is disconnected we have $\lambda_2 = d$.

We will see the proof of the latest theorem in the following lecture.