Lecture 4: Cheeger's Inequalities cont’d

In which we finish the proof of Cheeger’s inequalities.

It remains to prove the following statement.

**Lemma 1** Let \( y \in \mathbb{R}_+^V \) be a vector with non-negative entries. Then there is a \( t \in (0, \max_v\{y_v\}) \) such that
\[
\phi(\{v : y_v \geq t\}) \leq \sqrt{2R_L(y)}
\]

We will provide a probabilistic proof. Without loss of generality (multiplication by a scalar does not affect the Rayleigh quotient of a vector) we may assume that \( \max_v y_v = 1 \). We consider the probabilistic process in which we pick \( t > 0 \) in such a way that \( t^2 \) is uniformly distributed in \([0, 1]\) and then define the non-empty set \( S_t := \{v : y_v \geq t\} \).

We claim that
\[
\frac{\mathbb{E} d(S_t, V - S_t)}{\mathbb{E} \, d(S_t)} \leq \sqrt{2R_L(y)}
\]

Notice that Lemma 1 follows from such a claim, because of the following useful fact.

**Fact 2** Let \( X \) and \( Y \) be random variables such that \( \mathbb{P}[Y > 0] = 1 \). Then
\[
\mathbb{P} \left[ \frac{X}{Y} \leq \frac{\mathbb{E} X}{\mathbb{E} Y} \right] > 0
\]

**Proof:** Call \( r := \frac{\mathbb{E} X}{\mathbb{E} Y} \). Then, using linearity of expectation, we have \( \mathbb{E} X - rY = 0 \), from which it follows \( \mathbb{P}[X - rY \leq 0] > 0 \), but, whenever \( Y > 0 \), which we assumed to happen with probability 1, the conditions \( X - rY \leq 0 \) and \( \frac{X}{Y} \leq r \) are equivalent. \( \square \)

It remains to prove (1).

To bound the denominator, we see that
\[ \mathbb{E} d|S_t| = d \cdot \sum_{v \in V} \mathbb{P}[v \in S_t] = d \sum_v y_v^2 \]

because
\[ \mathbb{P}[v \in S_t] = \mathbb{P}[y_v \geq t] = \mathbb{P}[y_v^2 \geq t^2] = y_v^2 \]

To bound the numerator, we say that an edge is cut by \( S_t \) if one endpoint is in \( S_t \) and another is not. We have
\[
\mathbb{E} E(S_t, V - S_t) = \sum_{\{u, v\} \in E} \mathbb{P}[\{u, v\} \text{ is cut}]
\]
\[
= \sum_{\{u, v\} \in E} |y_v^2 - y_u^2| = \sum_{\{u, v\} \in E} |y_v - y_u| \cdot (y_u + y_v)
\]

Applying Cauchy-Schwarz, we have
\[
\mathbb{E} E(S_t, V - S_t) \leq \sqrt{\sum_{\{u, v\} \in E} (y_v - y_u)^2} \cdot \sqrt{\sum_{\{u, v\} \in E} (y_v + y_u)^2}
\]

and applying Cauchy-Schwarz again (in the form \((a + b)^2 \leq 2a^2 + 2b^2\)) we get
\[
\sum_{\{u, v\} \in E} (y_v + y_u)^2 \leq \sum_{\{u, v\} \in E} 2y_v + 2y_u^2 = 2d \sum_v y_v^2
\]

Putting everything together gives
\[
\frac{\mathbb{E} E(S_t, V - S_t)}{\mathbb{E} d|S_t|} \leq \sqrt{2 \sum_{\{u, v\} \in E} (y_v - y_u)^2 \cdot \frac{d}{d \sum_v y_v^2}}
\]

which is (1).