Lecture 4: Cheeger's Inequalities cont'd

In which we finish the proof of Cheeger's inequalities.

It remains to prove the following statement.

Lemma 1 *Let* $\mathbf{y} \in \mathbb{R}^V_{\geq 0}$ *be a vector with non-negative entries. Then there is a* $0 < t \leq \max_v\{y_v\}$ *such that*

$$\phi(\{v : y_v \geq t\}) \leq \sqrt{2R_L(\mathbf{y})}$$

We will provide a probabilistic proof. Without loss of generality (multiplication by a scalar does not affect the Rayleigh quotient of a vector) we may assume that $\max_v y_v = 1$. We consider the probabilistic process in which we pick $t > 0$ in such a way that t^2 is uniformly distributed in $[0, 1]$ and then define the non-empty set $S_t := \{v : y_v \geq t\}$.

We claim that

$$\frac{\mathbb{E}\, E(S_t, V - S_t)}{\mathbb{E}\, d|S_t|} \leq \sqrt{2R_L(\mathbf{y})} \tag{1}$$

Notice that Lemma 1 follows from such a claim, because of the following useful fact.

Fact 2 *Let* X *and* Y *be random variables such that* $\mathbb{P}[Y > 0] = 1$. *Then*

$$\mathbb{P}\left[\frac{X}{Y} \leq \frac{\mathbb{E}\, X}{\mathbb{E}\, Y}\right] > 0$$

PROOF: Call $r := \frac{\mathbb{E} X}{\mathbb{E} Y}$. Then, using linearity of expectation, we have $\mathbb{E}\, X - rY = 0$, from which it follows $\mathbb{P}[X - rY \leq 0] > 0$, but, whenever $Y > 0$, which we assumed to happen with probability 1, the conditions $X - rY \leq 0$ and $\frac{X}{Y} \leq r$ are equivalent. □

It remains to prove (1).

To bound the denominator, we see that

$$\mathbb{E}\, d|S_t| = d \cdot \sum_{v\in V} \mathbb{P}[v \in S_t] = d \sum_v y_v^2$$

because

$$\mathbb{P}[v \in S_t] = \mathbb{P}[y_v \geq t] = \mathbb{P}[y_v^2 \geq t^2] = y_v^2$$

To bound the numerator, we say that an edge is *cut* by S_t if one endpoint is in S_t and another is not. We have

$$\mathbb{E}\, E(S_t, V - S_t) = \sum_{\{u,v\}\in E} \mathbb{P}[\{u, v\} \text{ is cut}]$$

$$= \sum_{\{u,v\}\in E} |y_v^2 - y_u^2| = \sum_{\{u,v\}\in E} |y_v - y_u| \cdot (y_u + y_v)$$

Applying Cauchy-Schwarz, we have

$$\mathbb{E}\, E(S_t, V - S_t) \leq \sqrt{\sum_{\{u,v\}\in E} (y_v - y_u)^2} \cdot \sqrt{\sum_{\{u,v\}\in E} (y_v + y_u)^2}$$

and applying Cauchy-Schwarz again (in the form $(a + b)^2 \leq 2a^2 + 2b^2$) we get

$$\sum_{\{u,v\}\in E} (y_v + y_u)^2 \leq \sum_{\{u,v\}\in E} 2y_v + 2y_u^2 = 2d \sum_v y_v^2$$

Putting everything together gives

$$\frac{\mathbb{E}\, E(S_t, V - S_t)}{\mathbb{E}\, d|S_t|} \leq \sqrt{2\frac{\sum_{\{u,v\}\in E}(y_v - y_u)^2}{d \sum_v y_v^2}}$$

which is (1).