Midterm

To be finished individually. Due on Wednesday, February 20, 2013. Submit in class, or by email to trevisan at stanford dot edu

1. Let $G = (V, E)$ be a $d$-regular graph that is 3-colorable and such that there is a 3-coloring in which the color classes have equal size $|V|/3$. Let $A$ be the adjacency matrix and $L := 1 - \frac{1}{d} \cdot A$ be the Laplacian of $G$. Prove that $L$ has at least two eigenvalues which are greater than or equal to $3/2$, that is, $\lambda_{n-1} \geq 3/2$.

[Note: you get partial credit if you prove that there is a constant strictly bigger than 1, independent of $|V|$, such that two eigenvalues must be larger than that constant.]

Give an example in which the bound the tight.

Show that the converse is not true. (That is, give an example of a regular graph that is not 3-colorable but such that at least two eigenvalues of the normalized adjacency matrix are $\geq 3/2$.)

2. Recall that, given two graphs $G = (V, E_G)$ and $H = (V, E_H)$, consider the normalized non-uniform sparsest cut problem defined as follows:

$$nsc(G, H) := \min_{S \subseteq V} \frac{1}{|E_G|} \cdot \sum_{u,v} A_{u,v} |1_S(u) - 1_S(v)|$$

$$\frac{1}{|E_H|} \cdot \sum_{u,v} B_{u,v} |1_S(u) - 1_S(v)|$$

where $A$ is the adjacency matrix of $G$ and $B$ is the adjacency matrix of $H$, and the minimum is taken over all sets $S$ that are not empty and are different from $V$.

Consider the following continuos relaxation

$$\gamma(G, H) = \min_{x \in \mathbb{R}^V} \frac{1}{|E_G|} \cdot \sum_{u,v} A_{u,v} |x(u) - x(v)|^2$$

$$\frac{1}{|E_H|} \cdot \sum_{u,v} B_{u,v} |x(u) - x(v)|^2$$

Note that if $H$ is a clique with self-loops and $G$ is regular, then $\gamma(G, H) = \lambda_2(G)$ and $nsc(G, H) = nsc(G)$ is the normalized uniform sparsest cut problem on $G$. Recall also that $nsc(G) \leq 2\phi(G) \leq O(\sqrt{8\lambda_2})$, and so we may hope that,
say, when $G$ and $H$ are two arbitrary regular graphs, we have $nsc(G, H) \leq O(\sqrt{\gamma(G, H)})$.

Give a counterexample by showing (an infinite family of) regular graphs $G, H$ such that $nsc(G, H) \geq \Omega(1)$ but $\gamma(G, H) = o(1)$.

[Notes: you get full credit even if $G$ and $H$ are not regular. You should be able to get a family of graphs for which $\gamma(G, H) = O(1/n)$ and $nsc(G, H) = \Omega(1)$.

[Hint: Let $G$ be a cycle]