Practice Midterm

1. A directed graph $G = (V, E)$ is strongly connected if for any two vertices $u, v \in V$ there is a directed path in $G$ from $u$ to $v$. Let strong-CONN be the problem of deciding whether a given graph is strongly connected.

   (a) Show that strong-CONN is in $NL$.

   (b) Prove the $NL$-completeness of strong-CONN by giving a log-space reduction from ST-CONN to strong-CONN.

2. Suppose that there is a deterministic polynomial-time algorithm $A$ that on input (the description of) a circuit $C$ produces a number $A(C)$ such that

   $$\Pr_x[C(x) = 1] - \frac{2}{5} \leq A(C) \leq \Pr_x[C(x) = 1] + \frac{2}{5}.$$ 

   (a) Prove that it follows $P = BPP$.

   (b) Prove that there exists a deterministic algorithm $A'$ that, on input a circuit $C$ and a parameter $\epsilon$, runs in time polynomial in the size of $C$ and in $1/\epsilon$ and produces a value $A'(C, \epsilon)$ such that

   $$\Pr_x[C(x) = 1] - \epsilon \leq A'(C, \epsilon) \leq \Pr_x[C(x) = 1] + \epsilon.$$ 

   (c) Prove that there exists a deterministic algorithm $A''$ that, on input a circuit $C$ computing a function $f : \{0, 1\}^n \rightarrow \{1, \ldots, k\}$ and a parameter $\epsilon$, runs in time polynomial in the size of $C$, in $1/\epsilon$ and in $k$, and produces a value $A''(C, \epsilon)$ such that

   $$\mathbb{E}_x[f(x)] - \epsilon \leq A''(C, \epsilon) \leq \mathbb{E}_x[f(x)] + \epsilon.$$ 

   [For this question, you can think of $C$ as being a circuit with $\log k$ outputs, and the outputs of $C(x)$ are the binary representation of $f(x)$.

3. Prove that if $PSPACE \subseteq SIZE(poly(n))$, then $PSPACE$ is contained in the polynomial hierarchy.

   [Note: this may be harder than the other problems]

4. Consider a two-dimensional grid graph, that is, the graph whose set of vertices is $\{1, \ldots, \sqrt{n}\} \times \{1, \ldots, \sqrt{n}\}$ and such that a vertex $(i, j)$ has the four neighbors

   $$(i + 1 \mod \sqrt{n}, j), (i - 1 \mod \sqrt{n}, j), (i, j + 1 \mod \sqrt{n}), (i, j - 1 \mod \sqrt{n})$$

   Let $M$ be the transition matrix of this graph and let $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $M$. Prove that $\lambda_2 \leq 1 - \Omega(1/n)$.

   [Hint: recall how we prove that in an $n$-cycle $\lambda_2 \leq 1 - \Omega(1/n^2)$.]
5. A directed graph is $d$-regular if every vertex has in-degree $d$ and out-degree $d$. Prove that the $ST – CONN$ problem in directed regular graphs is in $L$.

[Hint: give a reduction to the undirected case.]