
Notes for Lecture 12

The Zig-Zag Graph Product

In this lecture we show that it is possible to “combine” a d -regular graph on D vertices and a D -regular graph on N vertices to obtain a d^2 -regular graph on ND vertices which is a good expander if the two starting graphs are.

Let the two starting graphs be denoted by H and G respectively. Then, the resulting graph, called the *zig-zag product* of the two graphs is denoted by $G\otimes H$.

Using $\bar{\lambda}_2(G)$ to denote the eigenvalue with the second-largest absolute value for a graph G , we shall prove that $\bar{\lambda}_2(G\otimes H) \leq \bar{\lambda}_2(G) + \bar{\lambda}_2(H) + (\bar{\lambda}_2(H))^2$.

1 Replacement product of two graphs

We first describe a simpler product for a “small” d -regular graph on D vertices (denoted by H) and a “large” D -regular graph on n vertices (denoted by G). Assume that for each vertex of G , there is some ordering on its D neighbors. Then we construct the replacement product (Figure 1) $G\odot H$ as follows:

- Replace each vertex of G with a copy of H (henceforth called a *cloud*). For $v \in V(G)$, $j \in V(H)$, let (v, j) is the j -th vertex in the cloud of v .
- Let $(u, v) \in E(G)$ be such that v is the i -th neighbor of u and u is the j -th neighbor of v . Then $((u, i), (v, j)) \in E(G\odot H)$. Also, if $(i, j) \in E(H)$, then $\forall v \in V(G) ((v, i), (v, j)) \in E(G\odot H)$.

Note that the replacement product constructed as above has nD vertices and is $(d+1)$ -regular.

2 Zig-zag product of two graphs

Given two graphs G and H as above, the zig-zag product $G\otimes H$ is constructed as follows (Figure 2):

- The vertex set $V(G\otimes H)$ is the same as in the case of the replacement product.
- $((u, i), (v, j)) \in E(G\otimes H)$ if there exist h and k such that $((u, i), (u, h)), ((u, h), (v, k))$ and $((v, k), (v, j))$ are in $E(G\otimes H)$ i.e. (v, j) can be reached from (u, i) by taking a step in the first cloud, then a step between the clouds and then a step in the second cloud (hence the name!).

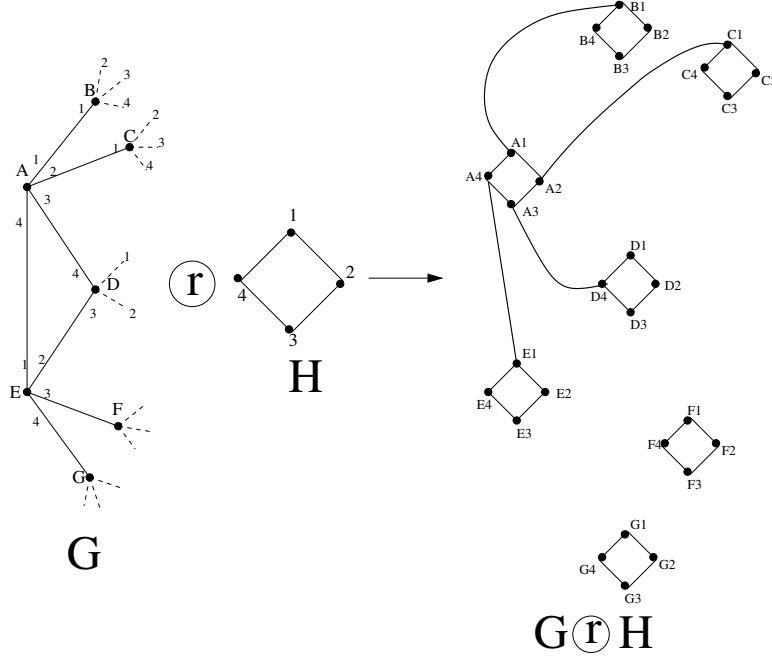


Figure 1: The replacement product of G and H (not all edges shown)

It is easy to see that the zig-zag product is a d^2 -regular graph on nD vertices. Let $M \in \mathbb{R}^{([n] \times [D]) \times ([n] \times [D])}$ be the transition matrix of $G \otimes H$. Using the fact that each edge in $G \otimes H$ is made up of three steps in $G \otimes H$, we can write M as BAB , where

$$B[(u, i), (v, j)] = \begin{cases} 0 & \text{if } u \neq v \\ \frac{1}{d} \cdot \# \text{ edges between } i \text{ and } j \text{ in } H & \text{if } u = v \end{cases}$$

$$A[(u, i), (v, j)] = \begin{cases} 1 & \text{if } v \text{ is the } i\text{-th neighbor of } u \text{ and } u \text{ is the } j\text{-th neighbor of } v \\ 0 & \text{otherwise} \end{cases}$$

Here B is the adjacency matrix of the replacement product after deleting all the edges between clouds and A is the adjacency matrix containing *only* the edges between clouds. Note that A is the adjacency matrix for a matching and is hence a permutation matrix.

3 Eigenvalues of the zig-zag graph

Theorem 1 *If G is a D -regular graph on N vertices and H is a d -regular graph on D vertices, then*

$$\bar{\lambda}_2(G \otimes H) \leq \bar{\lambda}_2(G) + \bar{\lambda}_2(H) + (\bar{\lambda}_2(H))^2 \quad (1)$$

We know that

$$\bar{\lambda}_2(G) = \max_{x \perp \mathbf{1}, \|x\|=1} |xMx^T| = \max_{x \perp \mathbf{1}} \frac{|xMx^T|}{xx^T}$$

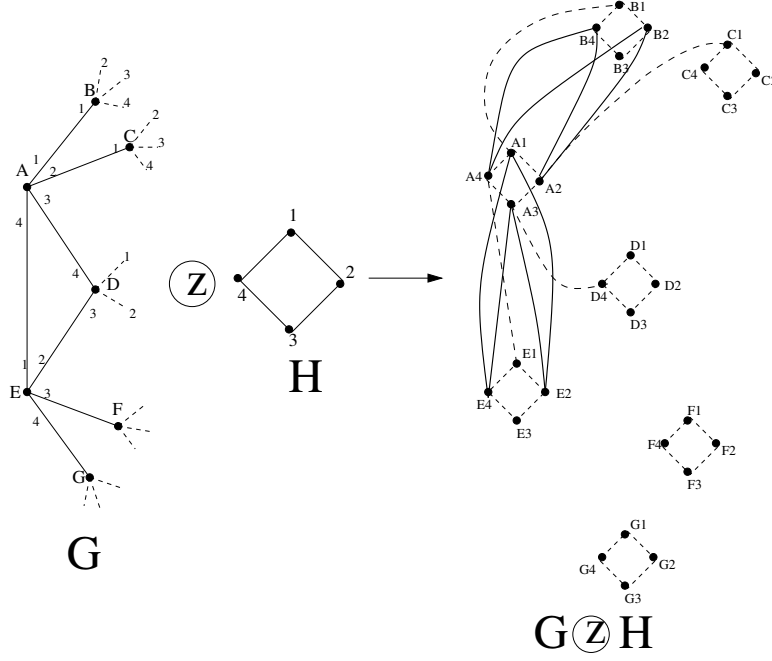


Figure 2: The zig-zag product of G and H and the underlying replacement product (not all edges shown)

Thus, it suffices to obtain a bound on the above expression for $G \otimes H$ when G and H are good expanders. To provide an intuition for the proof consider two extreme cases for a cut in $G \otimes H$. If the cut mostly includes or excludes entire clouds, then it can be viewed as a cut in G and the number of edges crossing it are almost the same as for the corresponding cut in G . If the cut splits almost all clouds in two parts, then one may think of it as N cuts in N copies of H . In both these cases then the number of edges crossing the cut will be “large” due the good expansion of G and H respectively. The following proof essentially breaks any vector x into the algebraic analogs of these two extremes.

PROOF: Given any vector $x \in \mathbb{R}^{ND}$, $x \perp \mathbf{1}$, one can write it as $x = x_{\parallel} + x_{\perp}$ where x_{\parallel} is constant on each cloud and x_{\perp} , restricted to any cloud is perpendicular to $\mathbf{1}^D$ (the all 1’s vector in D dimensions). In particular

$$x(u, i) := \frac{1}{D} \sum_j x(u, j)$$

$$x_{\perp}(u, i) = x(u, i) - x_{\parallel}(u, i)$$

We have

$$|xMx^T| = |xBABx^T| = |(x_{\parallel} + x_{\perp})BAB(x_{\parallel} + x_{\perp})|$$

$$\leq |x_{\parallel}BABx_{\parallel}^T| + 2|x_{\parallel}BABx_{\perp}^T| + |x_{\perp}BABx_{\perp}^T|$$

We now analyze each of these terms separately.

$$\begin{aligned}
|x_{\perp}BABx_{\perp}^T| &= |x_{\perp}BA(x_{\perp}B)^T| \\
&\leq \|x_{\perp}BA\| \cdot \|x_{\perp}B\| \quad (\text{by Cauchy - Schwarz}) \\
&= \|x_{\perp}B\| \cdot \|x_{\perp}B\| \quad (\text{since } A \text{ is a permutation matrix}) \\
&\leq \bar{\lambda}_2(H) \|x_{\perp}\| \cdot \bar{\lambda}_2(H) \|x_{\perp}\| \\
\Rightarrow |x_{\perp}BABx_{\perp}^T| &\leq \bar{\lambda}_2(H)^2 \|x_{\perp}\|^2 \tag{2}
\end{aligned}$$

In the above $\|x_{\perp}B\| \leq \bar{\lambda}_2(H) \|x_{\perp}\|$ follows from the fact that the restriction of x_{\perp} to any cloud is perpendicular to $\mathbf{1}^D$ and that B is a block-diagonal matrix whose action on the restriction is the same as that of the adjacency matrix of H . For the mixed term,

$$\begin{aligned}
|x_{\perp}BABx_{\parallel}^T| &= |x_{\perp}BA(x_{\parallel}B)^T| \\
&= |x_{\perp}BAx_{\parallel}^T| \quad (\because x_{\parallel} \text{ is parallel to } \mathbf{1}^D \text{ in each cloud}) \\
&\leq \|x_{\perp}B\| \cdot \|x_{\parallel}\| \\
&\leq \bar{\lambda}_2(H) \cdot \|x_{\perp}\| \cdot \|x_{\parallel}\| \\
&\leq \frac{1}{2} \bar{\lambda}_2(H) (\|x_{\perp}\|^2 + \|x_{\parallel}\|^2) \quad (\text{by Cauchy - Schwarz}) \\
\Rightarrow |x_{\perp}BABx_{\parallel}^T| &\leq \frac{1}{2} \bar{\lambda}_2(H) (\|x_{\parallel}\|^2 + \|x_{\perp}\|^2) = \frac{1}{2} \bar{\lambda}_2(H) \|x\|^2 \tag{3}
\end{aligned}$$

Let $y \in \mathbb{R}^N$ be the vector defined as $y(u) = \frac{1}{D} \sum_i x(u, i)$; note that $y(u) = x_{\parallel}(u, j)$ for all j , and so $\|x_{\parallel}\|^2 = D\|y\|^2$. Let C be the transition matrix of G . Then

$$\begin{aligned}
|x_{\parallel}BABx_{\parallel}^T| &= |x_{\parallel}Ax_{\parallel}^T| \\
&= \left| \sum_{u,i,v,j} x_{\parallel}(u, i) A(u, i, (v, j)) x_{\parallel}(v, j) \right| \\
&= D \left| \sum_{u,v} y(u) C(u, v) y(v) \right| \\
&= D |yCy^T| \\
&\leq D \bar{\lambda}_2(G) \|y\|^2 \\
&= \bar{\lambda}_2(G) \|x_{\parallel}\|^2 \\
\Rightarrow |x_{\parallel}BABx_{\parallel}^T| &\leq \bar{\lambda}_2(G) \|x_{\parallel}\|^2 \tag{4}
\end{aligned}$$

Note that $|yCy^T| \leq \bar{\lambda}_2(G) \|y\|^2$ follows from the fact that $y \cdot \mathbf{1} = \sum_i y(i) = \frac{1}{D} \sum_i \sum_j x(v_{ij}) = 0$.

Combining the above bounds gives

$$\begin{aligned}
|xBABx^T| &\leq \bar{\lambda}_2(G) \|x_{\parallel}\|^2 + \bar{\lambda}_2(H)^2 \|x_{\perp}\| + \bar{\lambda}_2(H) \|x\|^2 \\
\Rightarrow |xBABx^T| &\leq (\bar{\lambda}_2(G) + \bar{\lambda}_2(H)^2 + \bar{\lambda}_2(H)) \|x\|^2
\end{aligned}$$

Using the characterization of $\bar{\lambda}_2$, we have

$$\bar{\lambda}_2(G \otimes H) = \max_{x \perp \mathbf{1}, \|x\|=1} |xBABx^T| \leq \bar{\lambda}_2(G) + \bar{\lambda}_2(H)^2 + \bar{\lambda}_2(H)$$

□

4 Using the Zig-Zag Product to Construct Expanders

First, we state without proof the existence of graphs with good expansion properties. The proof is simple and it will be given in a later lecture.

Theorem 2 *For every p prime and $t \leq p$ there is an explicit construction of a p^2 regular graph $G_{p,t}$ with p^{t+1} vertices such that $\bar{\lambda}_2(G_{p,t}) \leq \frac{t}{p}$.*

We will use the following special case of the previous theorem.

Corollary 3 *There is a constant d such that a d -regular graph H with d^4 vertices exists that satisfies $\bar{\lambda}_2(H) \leq \frac{1}{5}$.*

In particular, we can apply the theorem with $p = 37$ and $t = 7$, so that the degree is $(37)^2 = 1369$.

Using the H from above we shall construct inductively a family of progressively larger graphs, all of which are d^2 -regular and have $\bar{\lambda}_2 \leq \frac{1}{2}$.

Let $G_0 := H^2$. For $k \geq 1$ let $G_{k+1} = (G_k^2) \otimes H$.

Theorem 4 *For each $k \geq 1$, G_k has degree d^2 and $\bar{\lambda}_2(G_k) \leq \frac{1}{2}$.*

PROOF: We shall proceed by induction.

For the base case: $G_0 = H^2$ is d^2 -regular and $\bar{\lambda}_2(H^2) = \frac{1}{25}$.

For the inductive step, assume the statement for k , *i.e.* G_k has degree d^2 and $\bar{\lambda}_2(G_k) \leq \frac{1}{2}$.

Then G_k^2 has degree $d^4 = |V(H)|$, so that the product $(G_k^2) \otimes H$ is defined. Moreover, $\bar{\lambda}_2(G_k^2) \leq \frac{1}{4}$. Applying the construction, we get that G_{k+1} has degree d^2 and

$$\bar{\lambda}_2(G_{k+1}) \leq \frac{1}{4} + \frac{1}{5} + \frac{1}{25} = \frac{49}{100} < \frac{1}{2}$$

This completes the proof. □

Finally note that G_k has $d^{4(k+1)}$ vertices.

5 References

The Zig-Zag graph product was defined and analysed by Reingold, Vadhan and Wigderson [RVW02]

References

- [RVW02] Omer Reingold, Salil Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. *Annals of Mathematics*, 155(1):157–187, 2002. 5