Notes for Lecture 28

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Summary

Today we define the notion of computational zero knowledge and show that the simulator we described in the last lecture establishes the computational zero knowledge property of the 3-coloring protocol.

1 The Protocol and the Simulator

Recall that we use a commitment scheme \((C, O)\) for messages in \(\{1, 2, 3\}\), and that the common input to the prover and the verifier is a graph \(G = ([n], E)\), where \([n] := \{1, 2, \ldots, n\}\). The prover, in addition, is given a valid 3-coloring \(\alpha : [n] \to \{1, 2, 3\}\) of \(G\).

The protocol is defined as follows:

- The prover picks a random permutation \(\pi : \{1, 2, 3\} \to \{1, 2, 3\}\) of the set of colors, and defines the 3-coloring \(\beta(v) := \pi(\alpha(v))\). The prover picks \(n\) keys \(K_1, \ldots, K_n\) for \((C, O)\), constructs the commitments \(c_v := C(K_v, \beta(v))\) and sends \((c_1, \ldots, c_n)\) to the verifier;
- The verifier picks an edge \((u, v) \in E\) uniformly at random, and sends \((u, v)\) to the prover;
- The prover sends back the keys \(K_u, K_v\);
- If \(O(K_u, c_u)\) and \(O(K_v, c_v)\) are the same color, or if at least one of them is equal to \(FAIL\), then the verifier rejects, otherwise it accepts.

For every verifier algorithm \(V^*\), we defined a simulator algorithm \(S^*\) which repeats the following procedure until the output is different from \(FAIL\):

**Algorithm** \(S^*_{1\text{round}}\)

- Input: graph \(G = ([n], E)\)
• Pick random coloring $\gamma : [n] \to \{1, 2, 3\}$.

• Pick $n$ random keys $K_1, \ldots, K_n$

• Define the commitments $c_i := C(K_i, \gamma(i))$

• Let $(u, v)$ be the 2nd-round output of $V^*$ given $G$ as input and $c_1, \ldots, c_n$ as first-round message

• If $\gamma(u) = \gamma(v)$, then output FAIL

• Else output $((c_1, \ldots, c_n), (u, v), (K_u, K_v))$

We want to show that this simulator construction establishes the computational zero knowledge property of the protocol, assuming that $(C, O)$ is secure. We give the definition of computational zero knowledge below.

**Definition 1 (Computational Zero Knowledge)** We say that a protocol $(P, V)$ for 3-coloring is $(t, \epsilon)$ computational zero knowledge with simulator overhead so($\cdot$) if for every verifier algorithm $V^*$ of complexity $\leq t$ there is a simulator $S^*$ of complexity $\leq$ so($t$) on average such that for every algorithm $D$ of complexity $\leq t$, every graph $G$ and every valid 3-coloring $\alpha$ we have

$$P[D(P(G, \alpha) \leftrightarrow V^*(G)) = 1] - P[D(S^*(G)) = 1] \leq \epsilon$$

**Theorem 2** Suppose that $(C, O)$ is $(2t + O(nr), \epsilon/(4 \cdot |E| \cdot n))$-secure and that $C$ is computable in time $\leq r$.

Then the protocol defined above is $(t, \epsilon)$ computational zero knowledge with simulator overhead at most $1.6 \cdot t + O(nr)$.

2 Proving that the Simulation is Indistinguishable

In this section we prove Theorem 2.

Suppose that the Theorem is false. Then there is a graph $G$, a 3-coloring $\alpha$, a verifier algorithm $V^*$ of complexity $\leq t$, and a distinguishing algorithm $D$ also of complexity $\leq t$ such that

$$P[D(P(G, \alpha) \leftrightarrow V^*(G)) = 1] - P[D(S^*(G)) = 1] \geq \epsilon$$

Let $2R_{u,v}$ be the event that the edge $(u, v)$ is selected in the second round; then
\begin{align*}
\epsilon \leq & \quad |\mathbb{P}[D(P(G, \alpha) \leftrightarrow V^*(G)) = 1] - \mathbb{P}[D(S^*(G)) = 1]| \\
= & \quad \left| \sum_{(u,v) \in E} \mathbb{P}[D(P(G, \alpha) \leftrightarrow V^*(G)) = 1 \land 2R_{u,v}] \
- \sum_{(u,v) \in E} \mathbb{P}[D(S^*(G)) = 1 \land 2R_{u,v}] \right| \\
\leq & \quad \left| \sum_{(u,v) \in E} |\mathbb{P}[D(P(G, \alpha) \leftrightarrow V^*(G)) = 1 \land 2R_{u,v}] 
- \mathbb{P}[D(S^*(G)) = 1 \land 2R_{u,v}]| \right| \
\leq & \quad \frac{\epsilon}{|E|} \quad (1)
\end{align*}

So there must exist an edge \((u^*, v^*) \in E\) such that

\[ |\mathbb{P}[D(P \leftrightarrow V^*) = 1 \land 2R_{u^*, v^*}] - \mathbb{P}[D(S^*) = 1 \land 2R_{u^*, v^*}]| \geq \frac{\epsilon}{|E|} \]

(We have omitted references to \(G, \alpha\), which are fixed for the rest of this section.)

Now we show that there is an algorithm \(A\) of complexity \(2t + O(nr)\) that is able to distinguish between the following two distributions over commitments to \(3n\) colors:

- **Distribution (1)** commitments to the \(3n\) colors 1, 2, 3, 1, 2, 3, \ldots, 1, 2, 3;
- **Distribution (2)** commitments to \(3n\) random colors

**Algorithm A:**

- Input: \(3n\) commitments \(d_{a,i}\) where \(a \in \{1, 2, 3\}\) and \(i \in \{1, \ldots, n\}\);
- Pick a random permutation \(\pi : \{1, 2, 3\} \rightarrow \{1, 2, 3\}\)
- Pick random keys \(K_{u^*}, K_{v^*}\)
- Construct the sequence of commitments \(c_1, \ldots, c_n\) by setting:
  - \(c_{u^*} := C(K_{u^*}, \pi(\alpha(u^*)))\)
  - \(c_{v^*} := C(K_{v^*}, \pi(\alpha(v^*)))\)
  - for every \(w \in [n] - \{u^*, v^*\}\), \(c_w := d_{\pi(\alpha(w)), w}\)
- If the 2nd round output of \(V^*\) given \(G\) and \(c_1, \ldots, c_n\) is different from \((u^*, v^*)\) output 0
First, we claim that

$$\mathbb{P}[A(\text{Distribution 1}) = 1] = \mathbb{P}[D(P \leftrightarrow V^*) = 1 \land 2R_{u^*,v^*}]$$  \hfill (2)

This follows by observing that $A$ on input Distribution 1 behaves exactly like the prover given the coloring $\alpha$, and that $A$ accepts if and only if the event $2R_{u^*,v^*}$ happens and $D$ accepts the resulting transcript.

Next, we claim that

$$|\mathbb{P}[A(\text{Distribution 2}) = 1] - \mathbb{P}[D(S^*) = 1 \land 2R_{u^*,v^*}]| \leq \frac{\epsilon}{2|E|}$$  \hfill (3)

To prove this second claim, we introduce, for a coloring $\gamma$, the quantity $DA(\gamma)$, defined as the probability that the following probabilistic process outputs 1:

- Pick random keys $K_1, \ldots, K_n$
- Define commitments $c_u := C(K_u, \gamma(u))$
- Let $(u, v)$ be the 2nd round output of $V^*$ given the input graph $G$ and first round message $c_1, \ldots, c_n$
- Output 1 iff $(u, v) = (u^*, v^*)$, $\gamma(u^*) \neq \gamma(v^*)$, and

$$D((c_1, \ldots, c_n), (u^*, v^*), (K_{u^*}, K_{v^*})) = 1$$

Then we have

$$\mathbb{P}[A(\text{Distribution 2}) = 1] = \sum_{\gamma: \gamma(u^*) \neq \gamma(v^*)} \frac{3}{2} \cdot \frac{1}{3^n} \cdot DA(\gamma)$$  \hfill (4)

Because $A$, on input Distribution 2, first prepares commitments to a coloring chosen uniformly at random among all $1/(6 \cdot 3^{n-2})$ colorings such that $\gamma(u^*) \neq \gamma(v^*)$ and then outputs 1 if and only if, given such commitments as first message, $V^*$ replies with $(u^*, v^*)$ and the resulting transcript is accepted by $D$.

We also have

$$\mathbb{P}[D(S^*) = 1 \land 2R_{u^*,v^*}] = \frac{1}{\mathbb{P}[S_{1\text{Round}}^* \neq \text{FAIL}]} \cdot \sum_{\gamma: \gamma(u^*) \neq \gamma(v^*)} \frac{1}{3^n} \cdot DA(\gamma)$$  \hfill (5)
To see why Equation (5) is true, consider that the probability that $S^*$ outputs a particular transcript is exactly $1/\mathbb{P}[S^*_{1\text{Round}} \neq\text{FAIL}]$ times the probability that $S^*_{1\text{Round}}$ outputs that transcript. Also, the probability that $S^*_{1\text{Round}}$ outputs a transcript which involves $(u^*, v^*)$ at the second round and which is accepted by $D()$ conditioned on $\gamma$ being the coloring selected at the beginning is $DA(\gamma)$ if $\gamma$ is a coloring such that $\gamma(u^*) \neq \gamma(v^*)$, and it is zero otherwise. Finally, $S^*_{1\text{Round}}$ selects the initial coloring uniformly at random among all possible $3^n$ coloring.

From our security assumption on $(C, O)$ and from Lemma 6 in Lecture 27 we have

$$\left| \mathbb{P}[S^*_{1\text{Round}} \neq\text{FAIL}] - \frac{2}{3} \right| \leq \frac{\epsilon}{4|E|} \tag{6}$$

and so the claim we made in Equation (3) follows from Equation (4), Equation (5), Equation (6) and the fact that if $p, q$ are quantities such that $\frac{3}{2}p \leq 1$, $\frac{1}{q} \cdot p \leq 1$, and $|q - \frac{2}{3}| \leq \delta \leq \frac{1}{6}$ (so that $q \geq 1/2$), then

$$\left| \frac{3}{2}p - \frac{1}{q}p \right| = \frac{3}{2} \cdot p \cdot \frac{1}{q} \cdot \left| q - \frac{2}{3} \right| \leq 2\delta$$

(We use the above inequality with $q = \mathbb{P}[S^*_{1\text{Round}} \neq\text{FAIL}]$, $\delta = \epsilon/4|E|$, and $p = \sum_{\gamma: \gamma(u^*) \neq \gamma(v^*)} \frac{1}{3^n} DA(\gamma)$.)

Having proved that Equation (3) holds, we get

$$|\mathbb{P}[A(\text{Distribution 1}) = 1] - \mathbb{P}[A(\text{Distribution 2}) = 1]| \geq \frac{\epsilon}{2|E|}$$

where $A$ is an algorithm of complexity at most $2t + O(nr)$. Now by a proof similar to that of Theorem 3 in Lecture 27, we have that $(C, O)$ is not $(2t + O(nr), \epsilon/(2|E|n))$ secure.