Problem 1

Assume for the sake of contradiction that $R$ recognizes $L$. Let $X_i$ be all strings of length $i$ in lexi order. Formally, $X_i = \text{sort}\{s \mid |s| = n\}$. Let $c_1$ be some constant such that for all $s$, $K(s) \leq |s| + c_1$.\footnote{this exists by textbook lemma}

```python
def more_complex_than (r, s) :: String -> String -> Bool
    if( R(r,s) == ACCEPT) return ACCEPT

def more_complex_than_set (r, S) :: String -> Set of Strings -> Bool
    for all s in S:
        fork more_complex_than (r, s)
    if (all threads ACCEPT) return ACCEPT

def find_incompressible_string (n) :: Int -> String
    for every s in $X_{n+c_1+1}$:
        fork more_complex_than_set (s, $X_n$)
    ans_n = first s to be accepted
    return ans_n
```

Let $c_2$ be the size of the program above. We now claim that for all $n$, we have $2 \cdot c_2 + 3 + \log n \geq K(\text{ans}_n) \geq n$, which is impossible for sufficiently large $n$.

- First, we prove that $\text{ans}_n$ must exist:
  - By textbook lemma, we know $\exists s \in X_{n+c+1} : K(s) \geq |s| = n + c + 1$.
  - By textbook lemma, we also know $\forall s \in X_n : K(s) \leq n + c$.
  - Thus, there is some string in $X_{n+c+1}$ which is more complex than all strings in $X_n$, and thus $\text{ans}_n$ must exist (because all of its calls to $R$ must terminate and accept).

- Next, we will prove that $2 \cdot c_2 + 3 \log n \geq K(\text{ans}_n)$. For this, consider $\text{bd}(\text{Prog})_{01}^{\leq 2c_2} \leq 2^{1+\log n}$

- Lastly, we prove that $K(\text{ans}_n) \geq n$. This follows from: $K(\text{ans}_n) \geq \max_{s \in X_n} K(s) \geq n$. 

Problem 2

Our general idea is:

1. Assume for the sake of contradiction that $R$ recognizes $L$.
2. Using $R$, show that $S = \{x \mid K(x) \geq |x|\}$ is recognizable.
3. Show this leads to a contradiction.

Statement 1 $\Rightarrow$ Statement 2

We define $R_S$, a recognizer for $S$, as follows:

$R_S(x) :$
run $R(x, x, |x|-1)$
accept iff $R$ accepts

Statement 2 $\Rightarrow$ Statement 3

$M(n):$
for $s \leftarrow$ sort \{all strings of length $n$\} 
run $R_S(s)$
ans$_n =$ first $s$ to be accepted
return ans$_n$

As in problem 1, we now get $2 * c_2 + 3 + \log n \geq K(\text{ans}_n) \geq n$, contradiction.

Problem 3

Suppose for the sake of contradiction that $f$ is unbounded.

def $S(n):$
for $x \leftarrow \Sigma^*$ in lexi order
if $(f(x) \geq n)$:
  return $x$

We note that by construction, $K(S(n)) \geq f(S(n)) \geq n$. However, it also has a description $bd(S) \begin{array}{c}01 \end{array} n$
of size $2 * c_2 + 3 + \log n$, contradiction.