Problem 1

• Solution 1

\{L \mid L \subseteq \Sigma^*\} is uncountable. The number of TMs is countable. Therefore, there is an undecidable subset of \(\Sigma^*\).

• Solution 2

Consider \(L = \{1^{E(M,x)} \mid M(x) \text{ halts} \}\) for some encoding \(E\) (for example, \(E(x,y) = \text{bin}(1 \circ \text{bd}(x) \circ 01 \circ y)\)).

Problem 2

• \(L\) has recognizer \(R \Rightarrow\) exists decider \(D\) s.t. \(L = \{x \mid \exists y : D(x,y) \text{ accepts} \}\):

\[D(x,y) = \text{does } R(x) \text{ accept within } y \text{ steps?}\]

• exists decider \(D\) s.t. \(L = \{x \mid \exists y : D(x,y) \text{ accepts} \} \Rightarrow\) \(L\) has recognizer \(R\):

\[R(x) = \begin{cases} \text{for } y \text{ in } \Sigma^{**} \text{ in lexi order:} \\ \, \text{if } D(x,y) \text{ accepts, accept} \end{cases}\]

Problem 3

• \(L\) is countable

- Let \(L_M = \{x \mid M(x) \in A\}\)
- Let \(L' = \{L_M \mid M \text{ is a TM}\}\).
- Note that \(L'\) is countable since the number of TMs is countable. We will now show that \(L \subseteq L'\).
- Take any \(B \in L\), there must be some \(M_B\) such that \(x \in B \iff M_B(x) \in A\). Thus, \(B = \{x \mid M_B(x) \in A\} = L_{M_B} \in L'\).

• \(U\) is uncountable

- Consider \(f(x) = 0 \circ x\).
- For a string \(x\) and a set \(S\), define \(x \circ S = \{x \circ y \mid y \in S\}\).
- Consider \(S = \{(0 \circ A) \cup (1 \circ X) \mid X \subseteq \Sigma^*\}\).
- It follows that \(\forall B \in S : \forall x : x \in A \iff f(x) \in B\). Thus, \(S \subseteq U\).
- Furthermore, note that since \(|S| = |2^{2^*}|\), \(S\) is uncountable.
Grading Rubric

- There’s basically 5 proofs: P1 (30), P2 one direction (15), P2 other direction (15), P3 countable (20), P3 uncountable (20).

- For each section:
  - Decide if solution is “basically correct” or “way off” (incorrect reduction; reducing in wrong direction; etc ...)
  - “Way off” solutions = 0 points
  - “Basically correct solutions” = start from full credit, deduct points as necessary for minor technical mistakes
  - When taking off points, provide a short (1-2 sentence) explanation for why points are being deducted.