Problem 1

Show that the language \( L = \{ \langle M \rangle : \text{there exists a polynomial } p(\cdot) \text{ such that, for every } x \in \{0, 1\}^*, M(x) \text{ halts in time } \leq p(|x|) \} \) is undecidable.
Problem 2

Define $L = \{ (M_1, M_2) : L(M_1) = \overline{L(M_2)} \}$.

Show that $L$ is not recognizable.
Problem 3

Two binary strings $x$ and $y$ are $\varepsilon$-close if $|x| = |y|$, and $x$ and $y$ differ in at most $\varepsilon \cdot |x|$ positions.

Prove that $Q(x) = \min\{K(y) | x, y \text{ are .0001-close}\}$ is not computable. You can use the fact that $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$. 
Problem 4
Show that for every constant $c$, there are strings $x, y$ such that $K(xy) > K(x) + K(y) + c$.

(Note: Don’t get hung up on this problem. It is quite hard and is meant as a challenge if the other problems are completed)
Problem 5

Consider the language consisting of \((G, K)\) pairs, where \(G\) is a directed graph that can be made acyclic by removing \(k\) edges. Formally, the feedback arc set problem is \(L = \{(G = (V, E), k) : \exists S \subseteq E, |S| = k, G - S\text{ is acyclic}\}\).

Prove that \(L\) is NP-complete.
Problem 6

Consider a crossword-puzzle game to be:

An \( m \times n \) matrix (the board), where each entry is

\( \ast \) (blank, can fill in a letter) or

\( \# \) (can’t fill in a letter)

This can give us our jagged crossword board we’re used to where we have to fill in the blanks.

Furthermore, we have an alphabet. Say, \( \Sigma = \{a, b, c\} \).

And finally we have a language \( L \subseteq \Sigma^* \) be some finite list of words.

The crossword problem is then: given a board, can we fill in each \( \ast \) with a letter from the alphabet such that

- Every column (read from top to bottom) is a sequence of words from \( L \), separated by one or more \( \# \)s
- Every row (read from left to right) is a sequence of words from \( L \), separated by one or more \( \# \)s

Show that the crossword problem is NP-complete.

[Hint: there is more than one way to prove this result. One possibility is to work from Vertex Cover; note that you don’t even need to use the \( \# \)s]