Problem Set 7

This problem set is due on Wednesday April 4, by 5:00pm.
Use the CS172 drop box.
Write your name and your student ID number on your solution. Write legibly. The description of your proofs should be as clear as possible (which does not mean long – in fact, typically, good clear explanations are also short.) Be sure to be familiar with the collaboration policy, and read the overview in the class homepage www.cs.berkeley.edu/~luca/cs172.

1. Let \( A \) and \( B \) be two languages. Then show that:
   (a) If \( A \) and \( B \) are in NP, then so are \( A \cup B \) and \( A \cap B \).
   (b) If \( A \) and \( B \) are NP-complete, then \( A \cup B \) and \( A \cap B \) need not be NP-complete.

2. Let \( U = \{ \langle M, x, \#^t \rangle \mid \text{NDTM } M \text{ accepts input } x \text{ within } t \text{ steps on at least one branch} \} \).
Show that \( U \) is NP-complete.

3. For a function \( g : \mathbb{N} \rightarrow \mathbb{N} \), we say a language \( L \) is in \( \text{SIZE}(g(n)) \) if there exists a family of circuits \( C_1, C_2, \ldots \) (with \( C_i \) having \( i \) inputs and one output) such that:
   - \( \forall n \in \mathbb{N} \) the size of \( C_n \) is at most \( g(n) \)
   - \( \forall x \in \{0,1\}^n \ x \in L \iff C_n(x) = 1 \).

   In the class we saw a proof that \( \text{SIZE}(2^{o(n)}) \subsetneq \text{SIZE}(2^n) \) i.e. for every large enough \( n \) there exists a function \( f : \{0,1\}^n \rightarrow \{0,1\} \) that is not computable by circuits of size \( 2^{o(n)} \). This problem asks you to show such a “separation result” for a smaller function. Show that \( \text{SIZE}(n^3/100 \log n) \subsetneq \text{SIZE}(n^3) \).

   Note: You may try (just for fun, this is not part of the homework) generalizing your proof to show that for all \( g : \mathbb{N} \rightarrow \mathbb{N} \) such that \( g(n) = o(2^n/\log n) \) and some constant \( C \), \( \text{SIZE}(g(n)/C \log g(n)) \subsetneq \text{SIZE}(g(n)) \).

   This result can actually be extended much further. One can even show that \( \text{SIZE}(g(n)−1) \subsetneq \text{SIZE}(g(n)) \) - every gate counts! But this might take a little longer than the spring break to prove - so you can skip that.