Solutions of homework 6

1. Sipser problem 5.9.

**Solution Outline: (20 points)**

Let $L$ be a Turing-recognizable language over alphabet $\Sigma$, and $M$ be a Turing machine that recognizes $L$. Consider the function $f$ that maps any $w \in \Sigma^*$ to $\langle M, w \rangle$. Clearly, $f$ is computable. Furthermore,

$$ w \in L \iff M \text{ accepts } w \iff \langle M, w \rangle \in A_{TM} \iff f(w) \in A_{TM} $$

Hence, $f$ is a mapping reduction of $L$ to $A_{TM}$. The result follows.

2. Sipser problem 5.12. Prove the result using Rice’s theorem: show that $S = L_C$, for a properly defined $C$, and show that $S$ is non-empty and does not contain all Turing machines.

**Solution Outline: (20 points)**

Take $C = \{ L \mid w \in L \iff w^R \in L \}$. $S$ contains the Turing machine that accepts only the string $00$ ($\{00\} \in C$) and does not contain the Turing machine that accepts only the string $01$ ($\{01\} \notin C$). Hence, $S$ is non-empty, and does not contain all Turing machines.

3. (a) Suppose $B$ is an undecidable language such that $B \leq_m \overline{B}$. Prove that neither $B$ nor $\overline{B}$ is Turing-recognizable.

(b) (Sipser problem 5.11) Give an example of an undecidable language $B$ where $B \leq_m \overline{B}$. (A correct example of such a language without proof will get zero credit; a complete solution should include an explicit mapping reduction from $B$ to $\overline{B}$ and a proof that $B$ is undecidable.)

**Solution Outline: (15,20 points)**

(a) Suppose on the contrary that $\overline{B}$ is Turing-recognizable. Then, since $B \leq_m \overline{B}$, $B$ is also Turing-recognizable. Now, we have both $\overline{B}$ and $B$ are Turing-recognizable, and thus $B$ is decidable, a contradiction. Hence, $\overline{B}$ is Turing-recognizable. The same argument shows that $\overline{B}$ is not Turing-recognizable, because $B \leq_m \overline{B}$ implies $\overline{B} \leq_m B$.

(b) Consider the language: $B = \{ w \mid w = 0x \text{ for some } x \in A_{TM} \text{ or } w = 1y \text{ for some } y \in A_{TM} \}$. Clearly, we have $\overline{B} = \{ w \mid w = \epsilon \text{ or } w = 0x \text{ for some } x \in A_{TM} \text{ or } w = 1y \text{ for some } y \in A_{TM} \}$. It is then easy to verify that the following map yields a mapping reduction from $B$ to $\overline{B}$:

$$ f(w) = \begin{cases} 1x & \text{if } w = 0x \\ 0x & \text{if } w = 1x \end{cases} $$

Furthermore, it is easy to see that $A_{TM}$ mapping reduces to $B$, so $B$ is undecidable.

**Solution Outline:** (25 points)

The same argument as that used in the proof of Theorem 6.10 (in Sipser) that Th(\(\mathcal{N}, +\)) is decidable applies here; except we need a different regular language for atomic formulas corresponding to \(<\). For that, consider

\[
\Sigma_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}
\]

A string of symbols in \(\Sigma_2\) gives 2 rows of 0s and 1s. Consider each row to be a binary number and let \(B = \{ w \in \Sigma_2^* \mid \text{the top row of } w \text{ is less than the bottom row of } w \}\). We can then construct a DFA for \(B\) as follows: take \(Q = \{ q_0, q_{\text{accept}}, q_{\text{reject}} \}\), \(F = \{ q_{\text{accept}} \}\), and take the transition function

\[
\delta(q_i, b) = \begin{cases} 
q_0 & \text{if } b \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \\
q_{\text{accept}} & \text{if } i = 0 \text{ and } b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } i = \text{accept} \\
q_{\text{reject}} & \text{if } i = 0 \text{ and } b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ or } i = \text{reject}
\end{cases}
\]