Solutions of homework 5


Solution Outline: (20 points)

Here’s an algorithm for the language $S$: on input $<M>$, reject if $<M>$ is not a valid encoding of a DFA. Otherwise, write $L = L(M)$ for ease of notation. First, construct a DFA $M^R$ for $L^R$ by converting the NFA for $L^R$ (from problem set 1) into a DFA. Next, construct a DFA $A$ for $L \triangle L^R$ by running $M$ and $M^R$ in parallel and accepting iff exactly one of the two DFAs accepts.\(^1\) Now, using the decider for $E_{DFA}$, accept if $A \in E_{DFA}$ and reject otherwise.

To show correctness, observe that $A \in E_{DFA}$ iff for all $w \in \Sigma^*$, either $w \in L$ and $w \in L^R$, or $w \notin L$ and $w \notin L^R$. The latter is equivalent to $M$ accepting $w^R$ whenever it accepts $w$.


Solution Outline: (20 points)

We may formalize the problem as the following language: $B = \{<M, w> | M$ is a TM that on input $w$ ever attempts to move its head left when its head is on the left-most tape cell.$\}$. To see that $B$ is undecidable, assume on the contrary that there exists some TM $R$ that decides $B$, and we use $R$ to construct a TM $S$ that decides $A_{TM}$:

$S = \text{“On input } <M, w>: \$

1. Construct TM $M'$ on input $w$, copies $w$ onto the tape one position right, and writes a special symbol $\Box$ on the first position. Then, $M'$ simulates $M$ on input $w$ starting from the second tape position, with two changes. First, if the head reads the symbol $\Box$, it moves right, and stays in the same state. Second, if $M$ enters an accept state, $M'$ enters a special state where the head just keeps moving to the left (past the left-most tape cell).

2. Run $R$ on input $<M', w>$.

3. Accept if $R$ accepts, and reject if $R$ rejects.”

3. (a) Prove that $E_{TM}$ is Turing-recognizable.

(b) Prove that $A_{TM}$ is not mapping reducible to $E_{TM}$.

Solution Outline: (15,10 points)

(a) We construct a Turing machine that recognizes $E_{TM}$ as in the construction of an enumerator for a Turing-recognizable language. On input $<M>$, for $i = 1, 2, \ldots$, simulate $M$ on all strings of length at most $i$ for $i$ steps, and accept if $M$ accepts any of these strings. Note that it follows from (a) and $E_{TM}$ being undecidable that $E_{TM}$ is not Turing-recognizable.

\(^1\)In particular, $Q_A = Q_M \times Q_{M^R}$, $\delta_A((q_1, q_2), \sigma) = (\delta_M(q_1, \sigma), \delta_{M^R}(q_2, \sigma))$ and $F_A = F_M \times F_{M^R} \cup F_{M^R} \times F_{M^R}$.
(b) Suppose on the contrary that $A_{TM}$ is mapping reducible to $E_{TM}$. Then, the same reduction shows that $\overline{A_{TM}}$ is mapping reducible to $\overline{E_{TM}}$. Since $\overline{E_{TM}}$ is Turing-recognizable, this means that $\overline{A_{TM}}$ is also Turing-recognizable (using Theorem 5.2 in the text), a contradiction (to Corollary 4.17).

Alternatively, we could use Corollary 5.23 to derive a contradiction to (a).

4. For each of the following languages, give a proof that it is undecidable or describe an algorithm to decide it.

(a) $L_1 = \{ \langle M \rangle \mid M \text{ is a Turing machine that rejects all inputs of even length} \}$.
(b) $L_2 = \{ \langle M \rangle \mid M \text{ is a Turing machine that halts on an empty input} \}$.
(c) $L_3 = \{ \langle M \rangle \mid \text{there is some input } x \in \{0,1\}^* \text{ such that } M \text{ accepts } x \text{ in less than 100 steps} \}$.

Solution Outline: (10,10,15 points)

(a) $L_1$ is undecidable. To see this, assume on the contrary that there exists some TM $R_1$ that decides $L_1$, and we use $R_1$ to construct a TM $S_1$ that decides $A_{TM}$:

$S_1 =$ “On input $\langle M, w \rangle$:

1. Construct TM $M_1$ that on input $x$, accept if $|x|$ is odd. If $|x|$ is even, it simulates $M$ on input $w$. If $M$ accepts $w$, $M_1$ enters the reject state. If $M$ rejects $w$, $M_1$ enters the accept state. If $M$ loops, $M_1$ also loops.
2. Run $R_1$ on input $\langle M_1 \rangle$.
3. Accept if $R_1$ accepts, and reject if $R_1$ rejects.”

Observe that if $M$ accepts $w$, then $M_1$ is a Turing machine that rejects all inputs of even length. If $M$ rejects or loops on input $w$, then $M_1$ is a Turing machine that for each input of even length, either loops or accepts.

(b) $L_2$ is undecidable. To see this, assume on the contrary that there exists some TM $R_2$ that decides $L_2$, and we use $R_2$ to construct a TM $S_2$ that decides $A_{TM}$:

$S_2 =$ “On input $\langle M, w \rangle$:

1. Construct TM $M_2$ that ignores its input and simulates $M$ on input $w$ and accept (and halt) if $M$ does. If $M$ rejects $w$, $M_2$ keeps moving right upon reading any input (thereby looping).
2. Run $R_2$ on input $\langle M_2 \rangle$.
3. Accept if $R_2$ accepts, and reject if $R_2$ rejects.”

Observe that if $M$ accepts $w$, then $M_2$ is a Turing machine that halts on an empty input. If $M$ rejects or loops on input $w$, then $M_2$ is a Turing machine that loops on an empty input.

(c) $L_3$ is decidable. First, observe that if $\langle M \rangle \in L_3$, then there exists some string $x$ of length at most 100 such that $M$ accepts $x$ in less than 100 steps. This is because $M$ cannot read beyond the 100th position of its input in less than 100 steps. Therefore, to check whether an input $\langle M \rangle$ is in $L_3$, it suffices to simulate $M$ over all strings of length at most 100 for at most 99 steps, and accept if $M$ accepts one of these strings, and reject otherwise.