Lecture 6

In which we finish the proof of exact reconstruction in the stochastic block model, and introduce a new semirandom model with an adversarial component.

We continue the proof of exact reconstruction in the stochastic block model (SBM), by proving that the optimum to the minimum bisection SDP we have studied so far is also the unique optimum.

1 Exact Reconstruction, Continued

Recall our setup for the stochastic block model: We construct a random graph $G = (V, E)$, with $|V| = n$. We partition $V$ into $S_1$ and $S_2$, with $|S_1| = |S_2| = \frac{n}{2}$. We place edges between vertices of the same $S_i$ with probability $p$, and edges between our partition sets with probability $q < p$.

We write $a = np^2$ and $b = nq^2$. Last lecture, we started proving the following theorem:

**Theorem 1** There is a constant $\beta$ such that given $G = (V, E)$ drawn from our SBM distribution, if $a - b > \beta \cdot \sqrt{\log n} \cdot \sqrt{a + b}$, then we can exactly reconstruct our bisection $V = (S_1, S_2)$ with high probability.

Recap: We began our proof the previous lecture by writing our SDP for minimum bisection.

$$\max \sum_{u,v} A_{u,v}(x_u, x_v)$$

subject to $\|x_v\|^2 = 1$ for all $v \in V$

$$\|\sum_v x_v\|^2 = 0$$

We define our intended solution $\{x_v\}_{v \in V}$ below, and the matrix $X$ by $X_{uv} = \langle x_u, x_v \rangle$:

$$x_v = \frac{1}{\sqrt{n}}[1, 1, \ldots, 1] \text{ if } v \in V_1 \text{ and } x_v = \frac{1}{\sqrt{n}}[-1, -1, \ldots, -1] \text{ if } v \in V_2$$

so $X_{u,v} = 1$ if $u, v$ are on the same side

$X_{u,v} = -1$ if $u, v$ are on different sides
Recall that $X = \chi \chi^T$, where $\chi$ is an indicator vector for the cut:

$$\chi \in \mathbb{R}^n : \chi_v = 1 \text{ if } v \in S_1, \chi_v = -1 \text{ if } v \in S_2$$

Let $a_v$ be the number of neighbors of $v$ on the same side of the bisection, and let $b_v$ be the number of neighbors of $v$ on the opposite. Then:

$$\text{Cost}(\{x_v\}_{v \in V}) = \sum_v (a_v - b_v)$$

We want to prove that $\{x_v\}_{v \in V}$ is an optimal solution to our SDP by showing that it is feasible, applying SDP duality and obtaining a dual solution of the same cost. Below is the dual:

$$\min \sum_{v=1}^n y_v$$

$$\text{subject to } \text{diag}(y_1, \ldots, y_n) + y_0 \cdot J \succeq A$$

We introduced the candidate dual solution $y_0 = \frac{a + b}{2}$ and $y_v = (a_v - b_v)$. Note that $\text{Cost}(y) = \text{Cost}(\{y_v\}_{v=0}^n) = \sum_{v=1}^n y_v = \sum_v (a_v - b_v)$, so if it is dual feasible then it witnesses that the rank-one primal solution corresponding to the hidden cut is optimal.

1.1 Proving that the certificate is dual feasible

Feasibility of the proposed dual certificate is equivalent to the positive definiteness condition

$$M := \text{diag}(y) + y_0 J - A \succeq 0.$$ 

We shall actually prove a stronger statement that will be useful for the complementary slackness part of the proof. We intend to show that (a) $M \chi = 0$ with probability 1, where again $\chi$ is the indicator of the cut and that (b) $x^T M x > 0$ for all nonzero $x \perp \chi$ with high probability.

The first statement (a) we shall directly verify below. The second (b) will follow from a matrix concentration argument: we shall show that apart from one eigenvalue of 0 corresponding to $\chi$, the eigenvalues of $\mathbb{E}[M]$ are all $a - b$ and we will then argue that with high probability $\|M - \mathbb{E}[M]\|_{op} \leq O\sqrt{\log n \sqrt{a + b}}$, which will yield the theorem when $a - b > c\sqrt{\log n \sqrt{a + b}}$ for a suitable absolute constant $c > 0$.

To verify that $M \chi = 0$ with probability 1, we compute

$$M \chi = \text{diag}(y) \cdot \chi - A \chi + J \chi$$

$$= \text{diag}(y) \cdot \chi - A \chi,$$

where we have used balance to deduce $J \chi = 0$. We now observe that

$$\left(\text{diag}(y) \cdot \chi\right)_i = (a_i - b_i) \chi_i,$$
while

$$(A\chi)_i = \sum_{j=1}^{n} A_{ij} \chi_j$$

$$= \chi_i \sum_{j=1}^{n} A_{ij} \chi_j$$

$$= (a_i - b_i) \chi_i,$$

where we have used the fact that $\chi_i \chi_j = 1$ if $i$ and $j$ are in the same community and $-1$ otherwise. Thus, diag$(y) \chi = A\chi$ and the claim follows.

On the other hand, we observe $\mathbb{E}[y_i] = \mathbb{E} [a_i - b_i] = a - b$ for $1 \leq i \leq n$. Thus,

$$\mathbb{E}[M] = (a - b) \cdot I + \frac{a + b}{n} \cdot J - \mathbb{E}[A]$$

$$= (a - b) \cdot I + \frac{a + b}{n} \cdot J - \frac{a + b}{n} \cdot J - \frac{a - b}{n} \cdot \chi\chi^T$$

$$= (a - b) \cdot I - \frac{a - b}{n} \cdot \chi\chi^T.$$

It is now clear that $\chi$ is in the nullspace of $\mathbb{E}[M]$ (as it must be since it is in the nullspace of $M$ with probability 1) and that all the other eigenvalues of $\mathbb{E}[M]$ are $a - b$, as claimed.

We now make the following claim without proof. The proof uses a matrix analog of the Chernoff bound (the “Matrix Bernstein inequality”).

**Lemma 2** With high probability over the choice of the graph,

$$||M - \mathbb{E}[M]|| \leq O(\sqrt{\log n}\sqrt{a + b})$$

The following Lemma is the core of our analysis. (Note that it is not a probabilistic statement.)

**Lemma 3** Suppose that $b - a > ||M - \mathbb{E}M||$. Then the solution $X = \chi\chi^T$ is the **unique** optimum for the minimum bisection SDP.

**Proof:** We begin by showing that $y_0, \ldots, y_n$ defined above is feasible for the dual, which implies that $\chi\chi^T$ is an optimal solution. Let $x$ be any vector, and write $x = \alpha \chi + y$, where $y$ is orthogonal to $\chi$. Then

$$x^T M x = (\alpha \chi + y)^T M (\alpha \chi + y)$$

$$= \alpha^2 \chi^T M \chi + 2\alpha y^T M \chi + y^T M y$$

$$= y^T M y$$

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where the last line uses the fact, which we established before, that $M\chi = 0$ for every graph. Since $y$ is orthogonal to $\chi$, we have

$$y^T (E M) y = (a - b) \cdot ||y||^2$$

and, by definition of spectral norm and the assumption of the lemma, we have that, if $y \neq 0$,

$$y^T M y \geq y^T (E M) y - ||M - E M|| \cdot ||y||^2 = (a - b - ||M - E M||) \cdot ||y||^2 > 0$$

We conclude that

$$x^T M x \geq 0$$

for every $x$, meaning that $M \succeq 0$, and that

$$x^T M x > 0$$

for every $x$ that is not a multiple of $\chi$.

From the fact that $M$ is PSD we have

$$\sum_v (a_v - b_v) = \text{Cost}(\{x_v\}_{v \in V}) = A \cdot X$$

$$\leq \left[ \text{diag}(a_1 - b_1, \ldots, a_n - b_n) + \frac{a + b}{n} \cdot J \right] \cdot X$$

$$\leq \text{diag}(a_1 - b_1, \ldots, a_n - b_n) \cdot X + \frac{a + b}{n} [J \cdot X]$$

$$= \sum_v y_v \cdot X_{v,v} + \frac{a + b}{n} \cdot \sum_{u,v} \langle x_{u,v}, x_{v,v} \rangle$$

$$= \sum_v y_v + \frac{a + b}{n} \cdot \sum_v x_v^2$$

$$= \sum_v (a_v - b_v)$$

Which implies that $\chi \chi^T$ is an optimal solution. Let now $X$ be any other optimal solution. Thus, all of the above inequalities are actually equalities, and we have:

$$A \cdot X = \left[ \text{diag}(a_1 - b_1, \ldots, a_n - b_n) + \frac{a + b}{n} \cdot J \right] \cdot X$$

which implies $[\text{diag}(a_1 - b_1, \ldots, a_n - b_n) + \frac{a + b}{n} \cdot J - A] \cdot X = M \cdot X = 0$

To show uniqueness of our solution $\chi \chi^T$, it suffices to show that the $X = \chi \chi^T$ is the only solution that satisfies $M \cdot X = 0$.

We showed that the assumption of the lemma imply that for all $x \perp \chi$:

$$x^T M x \geq (a - b) - O(\sqrt{\log n} \sqrt{a + b}) > 0$$
We can also write a PSD matrix as positive combinations of certain rank 1 matrices $z_iz_i^T$, so:

$$X = \sum_i \lambda_i z_i z_i^T$$ where $\lambda_i > 0$

$$M \cdot X = M \cdot \left( \sum_i \lambda_i z_i z_i^T \right) M = \sum_i \lambda_i (z_i^T M z_i)$$

The quantity $z_i^T M z_i$ will always be strictly positive, unless either $\lambda_i = 0$ or $z_i$ is parallel to $\chi$. Therefore, if $M \cdot X = 0$, we must have $X = \chi \chi^T$, which proves uniqueness of our solution. □

Putting everything together, we see that there is a constant $\beta$ such that, if $a - b > \beta \cdot \sqrt{\log n} \cdot \sqrt{a + b}$, then with high probability the unique optimum of the SDP is $\chi \chi^T$ and the algorithm of the previous lecture finds the hidden partition.