# FPTAS for #BIS with Degree Bounds on One Side

**Jingcheng Liu**[1] Pinyan Lu[2]

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# Warm-up

## Definition (Fully polynomial-time approximation scheme (FPTAS))

An FPTAS for a function $f(\cdot)$ is an algorithm with:

- *Input:* $x, \varepsilon > 0$
- *Output:* $\widetilde{f(x)}$ such that $(1-\varepsilon)f(x) \le \widetilde{f(x)} \le (1+\varepsilon)f(x)$.
- *Running time:* $O(\text{poly}(|x|, 1/\varepsilon))$.

## #BIS

*Instance.* A bipartite graph $G = (U \uplus V, E)$.

*Output.* The number of Independent Sets in the Bipartite graph $G$.

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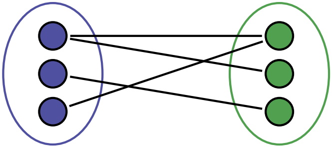

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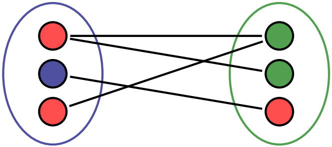

# Counting independent sets in general graphs

## Approximately count independent sets in $G$

- There is an FPTAS if the maximum degree $\Delta \leq 5$ [Weitz06], or *"average degree"* $\Delta \leq 5$. [SSY13]
- It is NP-hard if $\Delta \geq 6$, and even for $\Delta$-regular graph. [Sly10, SS12, GSV12]

## Independent set polynomial/Hardcore partition function

- Statistical physics.
- Graphical models.
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# Independent sets in bipartite graphs (#BIS)

## Why #BIS?

- A natural question by itself.
- As *a canonical* problem in classifications.
  Just like UNIQUE GAMES for optimization problems, or the END-OF-THE-LINE for fixed points and Nash equilibria.

Figure: A trichotomy for #CSP($\Gamma$). [DGJ10]

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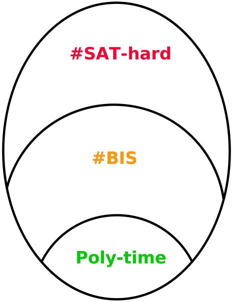


Figure: A trichotomy for $\#\mathrm{CSP}(\Gamma)$. [DGJ10]

# Conjectured Status of #BIS

## Conjecture [DGGJ00]

#BIS neither admits FPTAS/FPRAS, nor is as hard as #SAT to approximate.

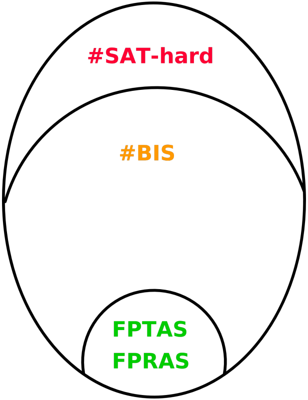


Figure: Conjectured status of #BIS. [DGGJ00]

# Some #BIS-equivalent problems

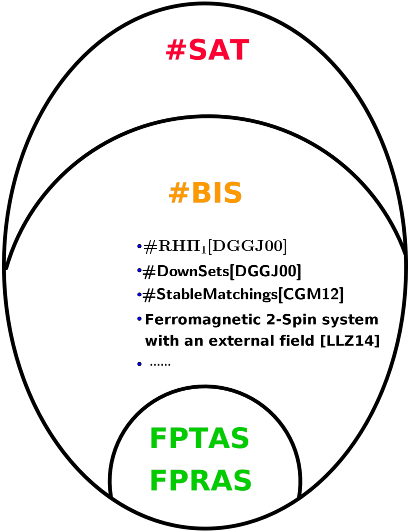


Figure: Some #BIS-equivalent problems from various domains.

# Known results

## Previous results

- Best known algorithm:
  Same as general graphs, namely FPTAS if $\Delta \leq 5$.
- Best known hardness:
  No NP-hardness known; As hard as general #BIS if $\Delta \geq 6$. [CGGGJSV14]

## Known Barriers

- Algorithmic: Same as for general graphs, a "global" phenomenon known as long-range correlation for higher degree.
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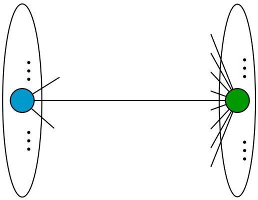
An FPTAS for #BIS with maximum degree bounds on one side, namely $\widetilde{\Delta} \le 5$.



$\widetilde{\Delta} \le 5$ Arbitrary

## Remark

- *Two sides often encode different objectives.*
- *Overcoming high degree barrier.*

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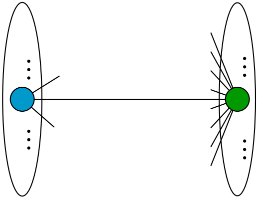

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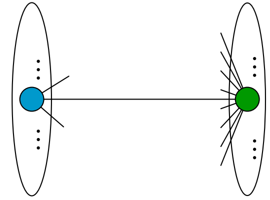
# Why high-degree vertex should NOT be a barrier

## Observation

- Consider a high-degree vertex, if we sample an independent set uniformly at random, it's more likely that one of its neighbors being occupied, than itself being occupied.
- Even if we ignore those vertices, the count is not changed by much.
- Caveat: Require the probability of its neighbors being occupied being not too small! Rescue: maximum degree on one side...



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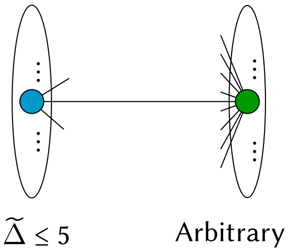

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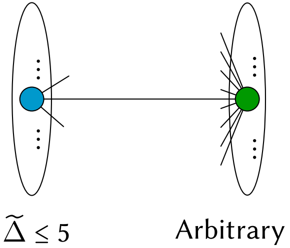
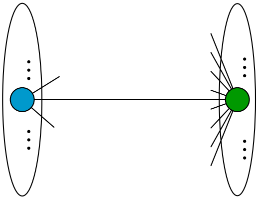


# One-sided maximum degree



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Arbitrary

Maximum degree on LHS

Upperbound of Pr[RHS is occupied]

deg of RHS

Lowerbound of Pr[LHS is occupied]

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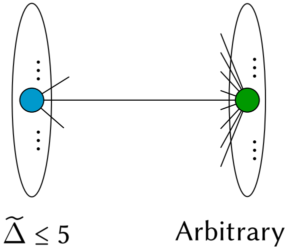


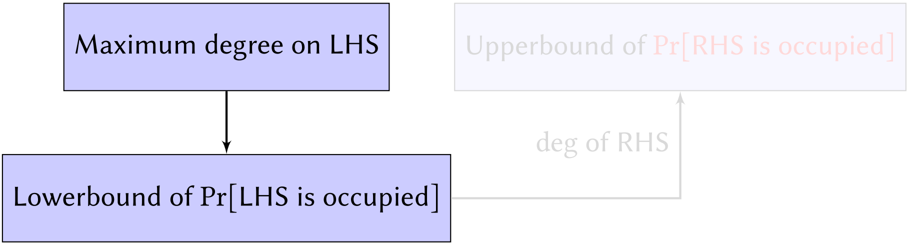

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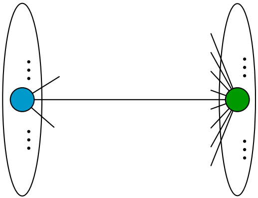

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Maximum degree on LHS

Lowerbound of Pr[LHS is occupied]

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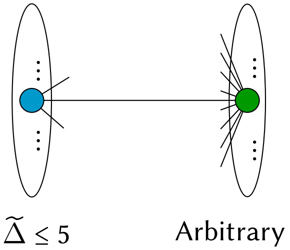
Upperbound of Pr[RHS is occupied]

# Alternative view: #BIS as a modified #CSP

## Alternative view as constraints

High-degree vertices are safe to ignore. They are "easy constraints".




We are going to view one side of vertices as variables, and the other side of vertices as constraints.

# Algorithm & Analysis

# Recurrence and Weitz's approach

Let $Z_G$ be the number of independent sets of $G$, and $R(G,u) \triangleq \frac{Z_G|_{u=1}}{Z_G|_{u=0}}$,

## Quick Fact by self-reducibility argument

For bipartite graph $G = (U \uplus V, E)$,

$$Z_G = 2^{|V|} \prod_{i=1}^{|U|} \left(1 + R(G_i, u_i)\right),$$

where $G_i = G \setminus \{u_j\}_{j=1}^{i-1}$.

## Theorem (Weitz'06)

$$R(G,u) = \prod_{i=1}^{\deg_G(u)} \frac{1}{1 + R(G_i, v_i)},$$

*where, if we let $N_G(u)$ be enumerated as $\{v_i\}_{i=1}^{d}$, then $G_i \triangleq (G-u) - \{v_j\}_{j=1}^{i-1}$.*

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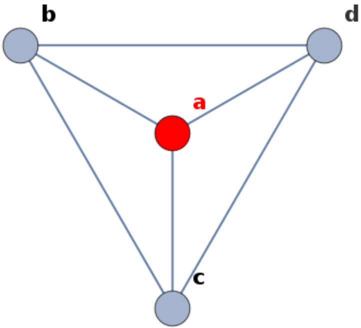
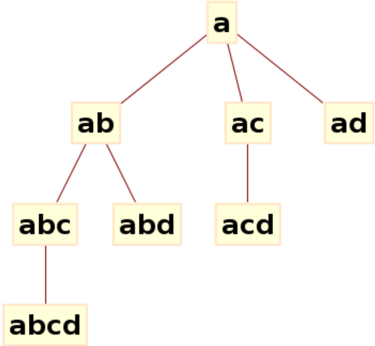
# The self-avoiding walk tree




(a) $K_4$.

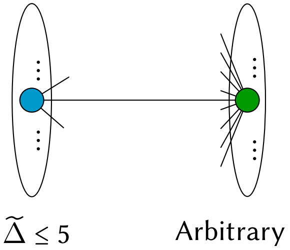



(b) SAW tree of $K_4$.

**Algorithm [Weitz06]**

Truncate the exponential-sized tree back to polynomial size.
If $\Delta \leq 5$, truncating at $\log(n/\varepsilon)$ depth gives good approximation.

# The 2-layer recurrence for modified #CSP




**Combine 2-layers of Weitz's, alternate between the two sides.**

Weitz's recurrence: $R(G,u) = \prod_{i=1}^{d} \left(1 + R(G_i, v_i)\right)^{-1}$,

2-layers recurrence: $R(G,u) = \prod_{i=1}^{d} \left(1 + \prod_{j=1}^{w_i} \left(1 + R\left(G_{i,j}, u_{i,j}\right)\right)^{-1}\right)^{-1}$.

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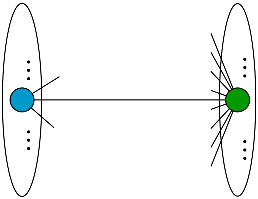


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# Exponential decay of correlations

**Algorithm**

Recursively compute $R$ and truncate at depth $L$. Call this estimate $R_L$.

**Key Lemma (Correlation decay)**

Let $G$ be a bipartite graph with maximum degree 5 on the LHS vertices, and $u$ be a vertex on the left. Then

$$|R_L(G,u) - R(G,u)| \le O(\alpha^L), \tag{1}$$

where $\alpha < 1$ is a fixed constant.

**Proof by induction?**

$$|R_{L+1}(G,u) - R(G,u)| \le \alpha \, |R_L(G,u) - R(G,u)| \, . \tag{2}$$

☹ Unfortunately this is not true, even for very simple graph families.

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# Amortized analysis

Consider all equivalent recurrences to $R(\cdot)$ in the form of

$$R^{\phi} \triangleq \phi \circ R \circ \phi^{-1}.$$

## Ideal recurrence

- We have step-wise contraction:

$$\left|R^{\phi}_{L+1}(G,u) - R^{\phi}(G,u)\right| \le \alpha \left|R^{\phi}_{L}(G,u) - R^{\phi}(G,u)\right|. \tag{3}$$

- Smaller Pr[RHS is occupied] should lead to better bounds on $\alpha$.

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# The final ingredient: adaptively truncate the tree

**Still a minor gap..**

The degree on the right-hand side could be abitrary!
Truncating at $\log(n/\varepsilon) \not\Longrightarrow$ a polynomial sized tree..

Pretend the SAW tree is a 45-ary branching tree, then do BFS with a budget of visiting at most $45^L$ nodes. Stop when run out of budget.

**Key Lemma (Correlation decay w.r.t modified depth)**

Let $G$ be a bipartite graph with maximum degree 5 on the LHS vertices, and $u$ be a vertex on the left. Then

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Finally we take $L \approx \log(n/\varepsilon)$, the algorithm will be an FPTAS.

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# Conclusion and Open Questions

## Conclusion

There is an FPTAS for #BIS with maximum degree 5 on one side.

## Open Problems:

- #BIS?
- How about a "one-sided" version for bipartite hard-core? (#BIS-hard)
- Better understanding of the limit of the correlation decay technique?
- Normal form for $\varphi \circ R \circ \varphi^{-1}$ ?