

FPTAS for #BIS with Degree Bounds on One Side

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Warm-up

Definition (Fully polynomial-time approximation scheme (FPTAS))

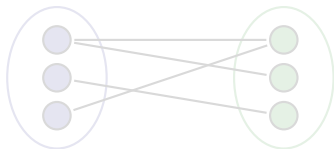
An FPTAS for a function $f(\cdot)$ is an algorithm with:

- *Input:* $x, \varepsilon > 0$
- *Output:* $\widetilde{f}(x)$ such that $(1 - \varepsilon)f(x) \leq \widetilde{f}(x) \leq (1 + \varepsilon)f(x)$.
- *Running time:* $O(\text{poly}(|x|, 1/\varepsilon))$.

#BIS

Instance. A bipartite graph $G = (U \uplus V, E)$.

Output. The number of Independent Sets in the Bipartite graph G .



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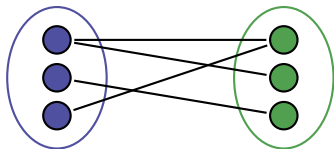
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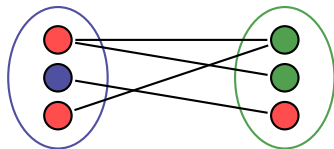
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Counting independent sets in general graphs

Approximately count independent sets in G

- There is an FPTAS if the maximum degree $\Delta \leq 5$ [Weitz06], or “average degree” $\Delta \leq 5$. [SSY13]
- It is NP-hard if $\Delta \geq 6$, and even for Δ -regular graph. [Sly10, SS12, GSV12]

Independent set polynomial/Hardcore partition function

- Statistical physics.
- Graphical models.
- Lovasz Local Lemma.
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Independent sets in bipartite graphs (#BIS)

Why #BIS?

- A natural question by itself.
- As *a canonical* problem in classifications.
Just like UNIQUE GAMES for optimization problems, or the END-OF-THE-LINE for fixed points and Nash equilibria.

Figure: A trichotomy for #CSP(Γ). [DGJ10]

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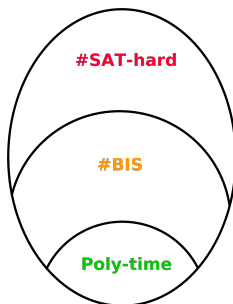


Figure: A trichotomy for $\#CSP(\Gamma)$. [DGJ10]

Conjectured Status of #BIS

Conjecture [DGGJ00]

#BIS neither admits FPTAS/FPRAS, nor is as hard as #SAT to approximate.

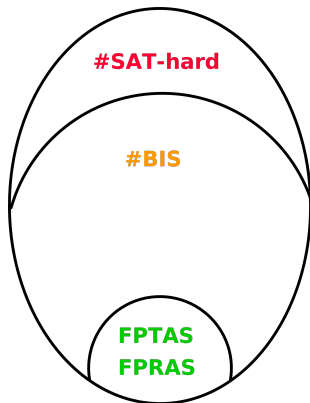


Figure: Conjectured status of #BIS. [DGGJ00]

Some #BIS-equivalent problems

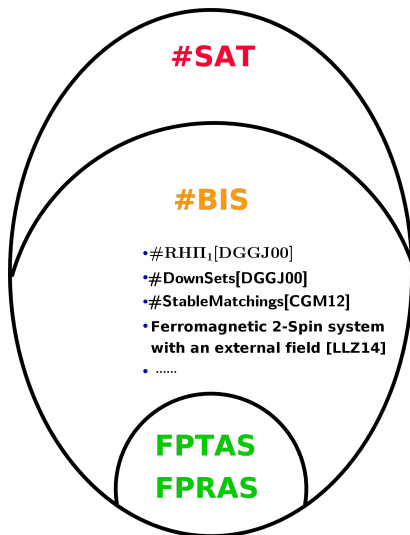


Figure: Some #BIS-equivalent problems from various domains.

Known results

Previous results

- Best known algorithm:
Same as general graphs, namely FPTAS if $\Delta \leq 5$.
- Best known hardness:
No NP-hardness known; As hard as general #BIS if $\Delta \geq 6$. [CGGGJSV14]

Known Barriers

- Algorithmic: Same as for general graphs, a “global” phenomenon known as long-range correlation for higher degree.
- Hardness: MAX-IND-SET is easy on bipartite graphs.

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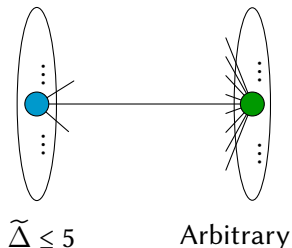
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An FPTAS for #BIS with maximum degree bounds on one side, namely $\tilde{\Delta} \leq 5$.



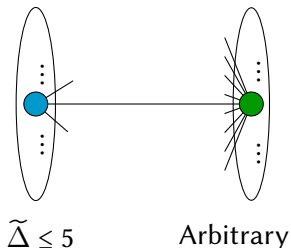
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- *Two sides often encode different objectives.*
- *Overcoming high degree barrier.*

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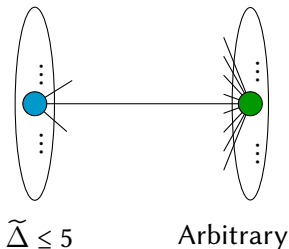
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Why high-degree vertex should NOT be a barrier

Observation

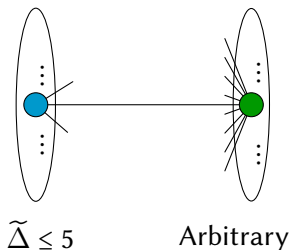
- Consider a high-degree vertex, if we sample an independent set uniformly at random, it's more likely that one of its neighbors being occupied, than itself being occupied.
- Even if we ignore those vertices, the count is not changed by much.
- **Caveat:** Require the probability of its neighbors being occupied being not too small! Rescue: maximum degree on one side...



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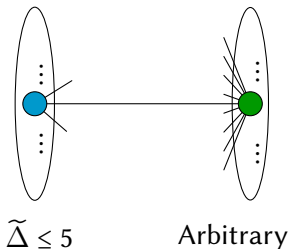
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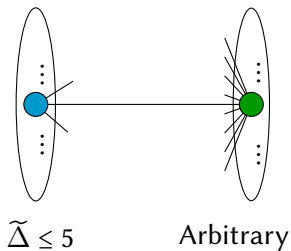
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One-sided maximum degree



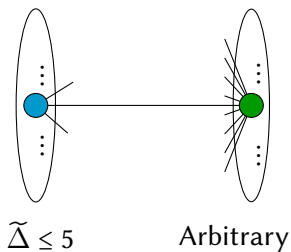
Maximum degree on LHS

Upperbound of $\Pr[\text{RHS is occupied}]$

Lowerbound of $\Pr[\text{LHS is occupied}]$

deg of RHS

One-sided maximum degree



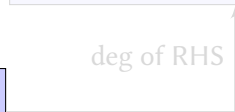
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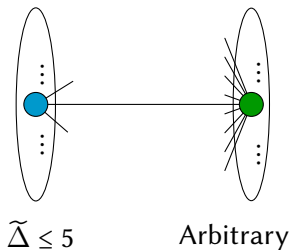
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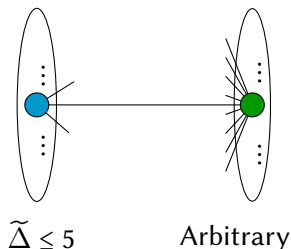
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Alternative view: #BIS as a modified #CSP

Alternative view as constraints

High-degree vertices are safe to ignore. They are “easy constraints”.



We are going to view one side of vertices as variables, and the other side of vertices as constraints.

Algorithm & Analysis

Recurrence and Weitz's approach

Let Z_G be the number of independent sets of G , and $R(G, u) \triangleq \frac{Z_G|_{u=1}}{Z_G|_{u=0}}$,

Quick Fact by self-reducibility argument

For bipartite graph $G = (U \uplus V, E)$,

$$Z_G = 2^{|V|} \prod_{i=1}^{|U|} (1 + R(G_i, u_i)),$$

where $G_i = G \setminus \{u_j\}_{j=1}^{i-1}$.

Theorem (Weitz'06)

$$R(G, u) = \prod_{i=1}^{\deg_G(u)} \frac{1}{1 + R(G_i, v_i)},$$

where, if we let $N_G(u)$ be enumerated as $\{v_i\}_{i=1}^d$, then $G_i \triangleq (G - u) - \{v_j\}_{j=1}^{i-1}$.

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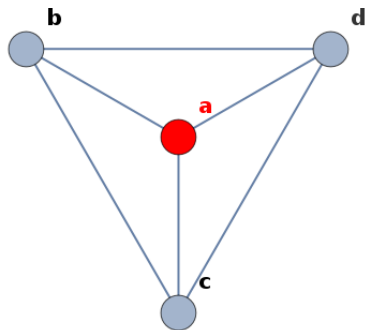
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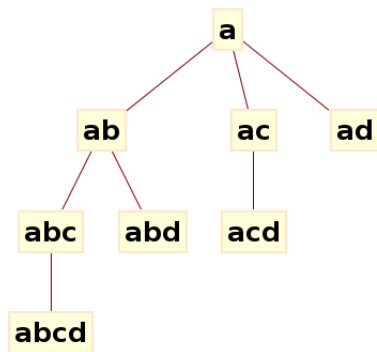
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The self-avoiding walk tree



(a) K_4 .



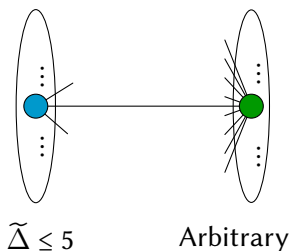
(b) SAW tree of K_4 .

Algorithm [Weitz06]

Truncate the exponential-sized tree back to polynomial size.

If $\Delta \leq 5$, truncating at $\log(n/\epsilon)$ depth gives good approximation.

The 2-layer recurrence for modified #CSP

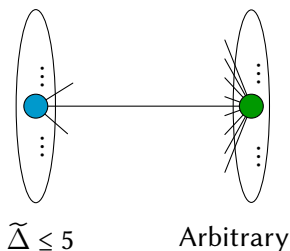


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Exponential decay of correlations

Algorithm

Recursively compute R and truncate at depth L . Call this estimate R_L .

Key Lemma (Correlation decay)

Let G be a bipartite graph with maximum degree 5 on the LHS vertices, and u be a vertex on the left. Then

$$|R_L(G, u) - R(G, u)| \leq O(\alpha^L), \quad (1)$$

where $\alpha < 1$ is a fixed constant.

Proof by induction?

$$|R_{L+1}(G, u) - R(G, u)| \leq \alpha |R_L(G, u) - R(G, u)|. \quad (2)$$

☹ Unfortunately this is not true, even for very simple graph families.

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Amortized analysis

Consider all equivalent recurrences to $R(\cdot)$ in the form of

$$R^\phi \triangleq \phi \circ R \circ \phi^{-1}.$$

Ideal recurrence

- We have step-wise contraction:

$$\left| R_{L+1}^\phi(G, u) - R^\phi(G, u) \right| \leq \alpha \left| R_L^\phi(G, u) - R^\phi(G, u) \right|. \quad (3)$$

- Smaller $\Pr[\text{RHS is occupied}]$ should lead to better bounds on α .

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The final ingredient: adaptively truncate the tree

Still a minor gap..

The degree on the right-hand side could be arbitrary!

Truncating at $\log(n/\varepsilon) \not\Rightarrow$ a polynomial sized tree..

Pretend the SAW tree is a 45-ary branching tree, then do BFS with a budget of visiting at most 45^L nodes. Stop when run out of budget.

Key Lemma (Correlation decay w.r.t modified depth)

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Conclusion and Open Questions

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There is an FPTAS for #BIS with maximum degree 5 on one side.

Open Problems:

- #BIS?
- How about a “one-sided” version for bipartite hard-core? (#BIS-hard)
- Better understanding of the limit of the correlation decay technique?
- Normal form for $\varphi \circ R \circ \varphi^{-1}$?