FPTAS for Counting Monotone CNF

Jingcheng Liu[1] Pinyan Lu[2]

[1]University of California, Berkeley

[2]Microsoft Research

Overview

1. Monotone Formulas

2. Approximate Counting

3. Correlation Decay Analysis
   - Recurrence and Weitz's approach
   - Adaptively truncating SAW tree
   - Exponential decay of correlations

Monotone CNF

Definition (Monotone CNF)

A CNF formula is *monotone* if every literal appears positively.

Remark (Equivalent definition)

A CNF formula φ (as a Boolean function over $\{0,1\}^n$) is monotone if $\forall \mathbf{x} \le \mathbf{y}, \varphi(\mathbf{x}) \le \varphi(\mathbf{y})$.

Monotone CNF

Definition (Monotone CNF)

A CNF formula is *monotone* if every literal appears positively.

Remark (Equivalent definition)

A CNF formula φ (as a Boolean function over $\{0,1\}^n$) is monotone if $\forall \mathbf{x} \le \mathbf{y}, \varphi(\mathbf{x}) \le \varphi(\mathbf{y})$.

Monotonicity

Example (Monotone increasing of covering and domination)

Let X be a cover; then any $Y \supseteq X$ is also a cover. E.g.:

- Vertex covers
- Edge covers
- Dominating sets
- Set covers

Example (Monotone decreasing of packing)

Let X be a packing; then any $Y \subseteq X$ is also a packing. E.g.:

- Independent sets
- Matching
- Set packings

Monotonicity

Example (Monotone increasing of covering and domination)

Let X be a cover; then any $Y \supseteq X$ is also a cover. E.g.:

- Vertex covers
- Edge covers
- Dominating sets
- Set covers

Example (Monotone decreasing of packing)

Let X be a packing; then any $Y \subseteq X$ is also a packing. E.g.:

- Independent sets
- Matching
- Set packings

Examples

Example (Set Cover)

Universe $U = \{x, y, z\}$, and sets $A = \{x, y\}$, $B = \{x\}$, $C = \{x, z\}$, $D = \{y, z\}$.
$F \subseteq \{A, B, C, D\}$ is a *set cover* if $\bigcup F = U$.

Sets are variables, and the elements are constraints. In monotone CNF:

$$\underbrace{(A \vee B \vee C)}_{\text{element } x} \wedge \underbrace{(A \vee D)}_{\text{element } y} \wedge \underbrace{(C \vee D)}_{\text{element } z}.$$

Definition (Degree of a variable)

The *degree* of a variable is the number of times it occurs in the formula.

Examples

Example (Set Cover)

Universe $U = \{x, y, z\}$, and sets $A = \{x, y\}$, $B = \{x\}$, $C = \{x, z\}$, $D = \{y, z\}$.
$F \subseteq \{A, B, C, D\}$ is a *set cover* if $\bigcup F = U$.

Sets are variables, and the elements are constraints. In monotone CNF:

$$\underbrace{(A \vee B \vee C)}_{\text{element } x} \wedge \underbrace{(A \vee D)}_{\text{element } y} \wedge \underbrace{(C \vee D)}_{\text{element } z}.$$

Definition (Degree of a variable)

The *degree* of a variable is the number of times it occurs in the formula.

Overview

1. Monotone Formulas
2. Approximate Counting
3. Correlation Decay Analysis
   - Recurrence and Weitz's approach
   - Adaptively truncating SAW tree
   - Exponential decay of correlations

What is Counting?

Interested in: the number of satisfying assignments to a monotone CNF formula.

Definition (Complexity class #P)

#P: *counting* problems whose *decision* problem is in *NP*.

Exact counting is generally hard (#P-complete).

Definition (Fully polynomial-time approximation scheme(FPTAS))

An algorithm is an *FPTAS* if, given as input a problem instance of size n with true answer N and a parameter $\epsilon \in (0,1]$, it ouputs a *multiplicative approximation* $\hat{N}$ in time $\mathrm{poly}\left(n, \frac{1}{\varepsilon}\right)$ such that

$$(1-\varepsilon)N \leq \hat{N} \leq (1+\varepsilon)N.$$

What is Counting?

Interested in: the number of satisfying assignments to a monotone CNF formula.

Definition (Complexity class #P)

#P: *counting* problems whose *decision* problem is in *NP*.

Exact counting is generally hard (#P-complete).

Definition (Fully polynomial-time approximation scheme(FPTAS))

An algorithm is an *FPTAS* if, given as input a problem instance of size n with true answer N and a parameter $\epsilon \in (0, 1]$, it ouputs a *multiplicative approximation* $\hat{N}$ in time $\mathrm{poly}\left(n, \frac{1}{\varepsilon}\right)$ such that

$$(1 - \varepsilon)N \leq \hat{N} \leq (1 + \varepsilon)N.$$

Approximate Counting

Exact and approximate counting problems arise in many applications, e.g.:

- **Combinatorics:** Graph polynomials
- **Statistical physics:** Partition functions of spin systems
- **Game theory:** Dynamics on social networks; Pricing in combinatorial prediction market design
- **Machine learning:** Inference in graphical models

Some Previous Results on Monotone #CSPs

Algorithmic side:

- **Independent set (or vertex cover):** FPTAS if the maximum degree (or connective constant) $\Delta \leq 5$ [Wei'06; SSY'13], and no efficient approximation when $\Delta \geq 6$ unless RP=NP [Sly'10].
- **Matching:** FPTAS if the maximum degree (or connective constant) is bounded by a constant [BGK+'07;SSSY'15], otherwise only an *FPRAS* is known [JS'89].
- **Edge cover:** FPTAS [LLL'14].
- **Dominating set:** FP for interval/trapezoid graphs [KOU'11], but #P-hard under AP-reductions even on bipartite graphs [GGL'14].

Hardness side:

Approximating Boolean *#CSP* is generally hard for non-monotone CSPs, and the classification is complete [DGJR'10].

But the approximability of *monotone* CSP is still open.

Some Previous Results on Monotone #CSPs

Algorithmic side:

- **Independent set (or vertex cover):** FPTAS if the maximum degree (or connective constant) $\Delta \leq 5$ [Wei'06; SSY'13], and no efficient approximation when $\Delta \geq 6$ unless RP=NP [Sly'10].
- **Matching:** FPTAS if the maximum degree (or connective constant) is bounded by a constant [BGK+'07;SSSY'15], otherwise only an *FPRAS* is known [JS'89].
- **Edge cover:** FPTAS [LLL'14].
- **Dominating set:** FP for interval/trapezoid graphs [KOU'11], but #P-hard under AP-reductions even on bipartite graphs [GGL'14].

Hardness side:

Approximating Boolean $\#CSP$ is generally hard for non-monotone CSPs, and the classification is complete [DGJR'10].
But the approximability of *monotone* CSP is still open.

Some Previous Results on Monotone #CSPs

Algorithmic side:

- **Independent set (or vertex cover):** FPTAS if the maximum degree (or connective constant) $\Delta \leq 5$ [Wei'06; SSY'13], and no efficient approximation when $\Delta \geq 6$ unless RP=NP [Sly'10].
- **Matching:** FPTAS if the maximum degree (or connective constant) is bounded by a constant [BGK+'07;SSSY'15], otherwise only an *FPRAS* is known [JS'89].
- **Edge cover:** FPTAS [LLL'14].
- **Dominating set:** FP for interval/trapezoid graphs [KOU'11], but #P-hard under AP-reductions even on bipartite graphs [GGL'14].

Hardness side:

Approximating Boolean $\#CSP$ is generally hard for non-monotone CSPs, and the classification is complete [DGJR'10].

But the approximability of *monotone* CSP is still open.

Some Previous Results on Monotone #CSPs

Algorithmic side:

- **Independent set (or vertex cover):** FPTAS if the maximum degree (or connective constant) $\Delta \leq 5$ [Wei'06; SSY'13], and no efficient approximation when $\Delta \geq 6$ unless RP=NP [Sly'10].
- **Matching:** FPTAS if the maximum degree (or connective constant) is bounded by a constant [BGK+'07;SSSY'15], otherwise only an *FPRAS* is known [JS'89].
- **Edge cover:** FPTAS [LLL'14].
- **Dominating set:** FP for interval/trapezoid graphs [KOU'11], but #P-hard under AP-reductions even on bipartite graphs [GGL'14].

Hardness side:

Approximating Boolean $\#CSP$ is generally hard for non-monotone CSPs, and the classification is complete [DGJR'10].

But the approximability of *monotone* CSP is still open.

Some Previous Results on Monotone #CSPs

Algorithmic side:

- **Independent set (or vertex cover):** FPTAS if the maximum degree (or connective constant) $\Delta \leq 5$ [Wei'06; SSY'13], and no efficient approximation when $\Delta \geq 6$ unless RP=NP [Sly'10].
- **Matching:** FPTAS if the maximum degree (or connective constant) is bounded by a constant [BGK+'07;SSSY'15], otherwise only an *FPRAS* is known [JS'89].
- **Edge cover:** FPTAS [LLL'14].
- **Dominating set:** FP for interval/trapezoid graphs [KOU'11], but #P-hard under AP-reductions even on bipartite graphs [GGL'14].

Hardness side:

Approximating Boolean $\#CSP$ is generally hard for non-monotone CSPs, and the classification is complete [DGJR'10].
But the approximability of *monotone* CSP is still open.

Our Results

Theorem (Main Theorem)

Counting monotone CNF admits an FPTAS if the maximum degree $\Delta \leq 5$.

Corollaries

FPTAS for counting:

- *set covers (or hitting sets) with maximum set size $d \leq 5$.*
- *hypergraph independent sets with maximum degree $\Delta \leq 5$.*
- *hypergraph edge covers with hypergraph edge size at most 5.*
- *dominating sets in graphs with maximum degree $\Delta \leq 4$.*

* All of the above, except for dominating sets, are tight unless $RP = NP$.

Theorem

Counting 3D matchings admits an FPTAS if the maximum degree $\Delta \leq 4$.

Overview

1. Monotone Formulas

2. Approximate Counting

3. Correlation Decay Analysis
   - Recurrence and Weitz's approach
   - Adaptively truncating SAW tree
   - Exponential decay of correlations

Review of Weitz's approach

Let Z_G be the independent set polynomial. (e.g. number of independent sets)

Quick Fact

- Equivalent to computing $R(G, u) \triangleq \frac{Z_G|_{u=1}}{Z_G|_{u=0}}$.
- An additive approx. of R translates to a multiplicative approx. of Z_G.

Theorem (Weitz'06)

$$R(G, u) = R(T_{SAW}(G), u).$$

Sketch of proof: based on a recursive decomposition:

$$R(G, u) = \prod_{i=1}^{d} \frac{1}{1 + R(G_i, v_i)},$$

where $d \triangleq \deg_G(u)$, *and let* $N_G(u)$ *be enumerated as* $\{v_i\}_{i=1}^{d}$, *then* $G_i \triangleq (G - u) - \{v_j\}_{j=1}^{i-1}$.

Review of Weitz's approach

Let Z_G be the independent set polynomial. (e.g. number of independent sets)

Quick Fact

- Equivalent to computing $R(G,u) \triangleq \frac{Z_G|_{u=1}}{Z_G|_{u=0}}$.
- An additive approx. of R translates to a multiplicative approx. of Z_G.

Theorem (Weitz'06)

$$R(G,u) = R(T_{SAW}(G), u).$$

Sketch of proof: based on a recursive decomposition:

$$R(G,u) = \prod_{i=1}^{d} \frac{1}{1 + R(G_i, v_i)},$$

where $d \triangleq \deg_G(u)$*, and let* $N_G(u)$ *be enumerated as* $\{v_i\}_{i=1}^{d}$*, then* $G_i \triangleq (G - u) - \{v_j\}_{j=1}^{i-1}$.

Review of Weitz's approach

Let Z_G be the independent set polynomial. (e.g. number of independent sets)

Quick Fact

- Equivalent to computing $R(G, u) \triangleq \frac{Z_G|_{u=1}}{Z_G|_{u=0}}$.
- An additive approx. of R translates to a multiplicative approx. of Z_G.

Theorem (Weitz'06)

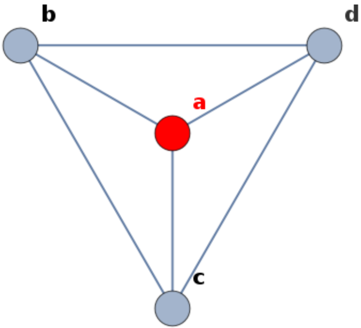
$$R(G, u) = R(T_{SAW}(G), u).$$

Sketch of proof: based on a recursive decomposition:

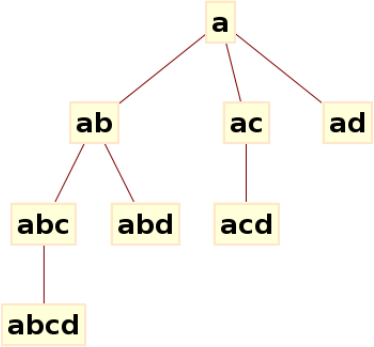
$$R(G, u) = \prod_{i=1}^{d} \frac{1}{1 + R(G_i, v_i)},$$

where $d \triangleq \deg_G(u)$*, and let* $N_G(u)$ *be enumerated as* $\{v_i\}_{i=1}^{d}$*, then* $G_i \triangleq (G - u) - \{v_j\}_{j=1}^{i-1}$.

Self-avoiding walk tree



(a) K_4.



(b) SAW tree of K_4.

Algorithm (Weitz'06)

Truncate the exponential-sized tree back to polynomial size. When $\Delta \leq 5$, truncating at $\log(n/\varepsilon)$ depth gives good approximation.

Recurrence for hypergraphs

Question

For hypergraphs, or monotone CNF formulas C, do we still have

$$R(C, x) = R(T_{SAW}(C), x)?$$

Here recall $R(C, x) \triangleq \frac{Z_C|_{x=0}}{Z_C|_{x=1}}$.

Unfortunately the most natural generalization of T_{SAW} to hypergraphs doesn't work.

Theorem (Sketch)

$$R(C, x) = R'\left(T_{SAW}(B(C)), x\right),$$

where

- *$B(C)$ is the bipartite graph representation of the hypergraph,*
- *R' is a two-layer recurrence that alternates between the two partites.*

Recurrence for hypergraphs

Question

For hypergraphs, or monotone CNF formulas C, do we still have

$$R(C, x) = R(T_{SAW}(C), x)?$$

Here recall $R(C, x) \triangleq \frac{Z_C|_{x=0}}{Z_C|_{x=1}}$.

Unfortunately the most natural generalization of T_{SAW} to hypergraphs doesn't work.

Theorem (Sketch)

$$R(C, x) = R'\left(T_{SAW}(B(C)), x\right),$$

where

- $B(C)$ *is the bipartite graph representation of the hypergraph,*
- R' *is a two-layer recurrence that alternates between the two partites.*

Recurrence for hypergraphs

We have the following *two-layer* recursive decomposition:

Theorem (Formal)

$$R(C, x) = \prod_{j=1}^{d} \left(1 - \prod_{i=1}^{w_j} \frac{R(C_{j,i}, x_{j,i})}{1 + R(C_{j,i}, x_{j,i})} \right),$$

where $d \triangleq \deg_x(C)$, *and the* d *clauses containing* x *be enumerated as* $\{c_j\}_{j=1}^{d}$. *Denote* $w_j \triangleq |c_j| - 1$, $C_j \triangleq \left(C \setminus \{c_k\}_{k \neq j}\right) \cup \{c_k \setminus x \mid j+1 \leq k \leq d\}$. *Let* $\{x_{j,i}\}_{i=1}^{w_j}$ *be the set of variables in* $c_j \setminus x$, *and* $C_{j,i} \triangleq (C_j \setminus c_j) \setminus \{x_{j,k}\}_{k=1}^{i-1}$.

Truncating the tree of exponential size?

- Truncate at $L \approx \log(n/\varepsilon)$ depth?
  ☹ The degree of the tree is unbounded! (i.e. w_j is unbounded)
- Truncate more *adaptively*?

Recurrence for hypergraphs

We have the following *two-layer* recursive decomposition:

Theorem (Formal)

$$R(C,x) = \prod_{j=1}^{d}\left(1 - \prod_{i=1}^{w_j} \frac{R(C_{j,i},x_{j,i})}{1+R(C_{j,i},x_{j,i})}\right),$$

where $d \triangleq \deg_x(C)$*, and the* d *clauses containing* x *be enumerated as* $\{c_j\}_{j=1}^{d}$*. Denote* $w_j \triangleq |c_j| - 1$, $C_j \triangleq \left(C \setminus \{c_k\}_{k\neq j}\right) \cup \{c_k \setminus x \mid j+1 \leq k \leq d\}$*. Let* $\{x_{j,i}\}_{i=1}^{w_j}$ *be the set of variables in* $c_j \setminus x$*, and* $C_{j,i} \triangleq (C_j \setminus c_j) \setminus \{x_{j,k}\}_{k=1}^{i-1}$*.*

Truncating the tree of exponential size?

- Truncate at $L \approx \log(n/\varepsilon)$ depth?
  ☹ The degree of the tree is unbounded! (i.e. w_j is unbounded)
- Truncate more *adaptively*?

Adaptively Truncated Tree

Pretend the tree is a 4-ary branching tree, then do BFS with a budget of visiting at most 4^L nodes. Stop when run out of budget.

Algorithm

Let $L_j = \max\left(0, L - \lceil \log_4(w_j + 1) \rceil\right)$.

$$R(C, x, L) = \prod_{j=1}^{d} \left(1 - \prod_{i=1}^{w_j} \frac{R\left(C_{j,i},\ x_{j,i},\ L_j\right)}{1 + R\left(C_{j,i},\ x_{j,i},\ L_j\right)}\right).$$

Now we take $L \approx \log(n/\varepsilon)$.

But is this a good approximation?..

Pretend the tree is a 4-ary branching tree, then do BFS with a budget of visiting at most 4^L nodes. Stop when run out of budget.

Algorithm

Let $L_j = \max\left(0, L - \lceil \log_4(w_j + 1) \rceil\right)$.

$$R(C, x, L) = \prod_{j=1}^{d} \left(1 - \prod_{i=1}^{w_j} \frac{R\left(C_{j,i},\ x_{j,i},\ L_j\right)}{1 + R\left(C_{j,i},\ x_{j,i},\ L_j\right)}\right).$$

Now we take $L \approx \log(n/\varepsilon)$.

But is this a good approximation?..

Exponential decay of correlations

Lemma (Correlation decay)

Let $\alpha = 0.981$, C be a monotone CNF formula with maximum degree 5, and x be a variable. Then

$$|R(C, x, L) - R(C, x)| \leq 5\sqrt{6}\alpha^L. \tag{1}$$

Proof by induction?

$$|R(C, x, L+1) - R(C, x)| \leq \alpha |R(C, x, L) - R(C, x)|. \tag{2}$$

☹ Unfortunately this is not true, even for very simple graph families.

FIX: amortized analysis

Let $\varphi(x) \triangleq 2\sinh^{-1}\left(\sqrt{x}\right)$ be a potential function, we analyze $\varphi \circ R \circ \varphi^{-1}$.

Exponential decay of correlations

Lemma (Correlation decay)

Let $\alpha = 0.981$*,* C *be a monotone CNF formula with maximum degree* 5*, and* x *be a variable. Then*

$$|R(C,x,L) - R(C,x)| \le 5\sqrt{6}\alpha^L. \tag{1}$$

Proof by induction?

$$|R(C,x,L+1) - R(C,x)| \le \alpha \left|R(C,x,L) - R(C,x)\right|. \tag{2}$$

☹ Unfortunately this is not true, even for very simple graph families.

FIX: amortized analysis

Let $\varphi(x) \triangleq 2\sinh^{-1}\left(\sqrt{x}\right)$ be a potential function, we analyze $\varphi \circ R \circ \varphi^{-1}$.

Exponential decay of correlations

Lemma (Correlation decay)

Let $\alpha = 0.981$, *C be a monotone CNF formula with maximum degree* 5, *and x be a variable. Then*

$$|R(C, x, L) - R(C, x)| \leq 5\sqrt{6}\alpha^L. \tag{1}$$

Proof by induction?

$$|R(C, x, L+1) - R(C, x)| \leq \alpha \, |R(C, x, L) - R(C, x)|\,. \tag{2}$$

☹ Unfortunately this is not true, even for very simple graph families.

FIX: amortized analysis

Let $\varphi(x) \triangleq 2\sinh^{-1}\left(\sqrt{x}\right)$ be a potential function, we analyze $\varphi \circ R \circ \varphi^{-1}$.

Exponential decay of correlations

Lemma (Correlation decay)

Let $\alpha = 0.981$, C *be a monotone CNF formula with maximum degree* 5, *and* x *be a variable. Then*

$$|R(C,x,L) - R(C,x)| \le 5\sqrt{6}\alpha^L. \tag{1}$$

Proof by induction?

$$|R(C,x,L+1) - R(C,x)| \le \alpha\,|R(C,x,L) - R(C,x)|\,. \tag{2}$$

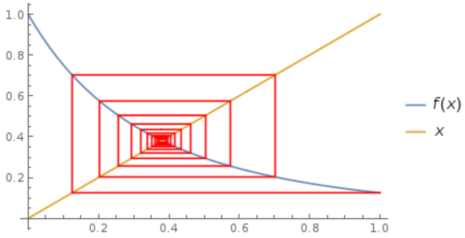
☹ Unfortunately this is not true, even for very simple graph families.

FIX: amortized analysis

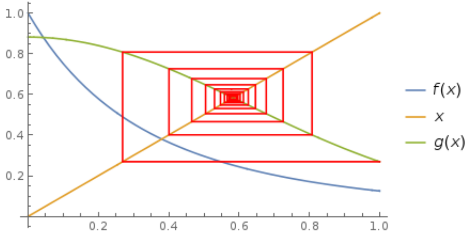
Let $\varphi(x) \triangleq 2\sinh^{-1}\left(\sqrt{x}\right)$ be a potential function, we analyze $\varphi \circ R \circ \varphi^{-1}$.

Exponential decay of correlations

Consider a very special case: the graph is a symmetric, infinite d-ary tree.
The recursion: $f(x) = \left(\frac{1}{1+x}\right)^d$. Amortized: $g(x) = \varphi\left(f(\varphi^{-1}(x))\right)$.



(a) Original function.



(b) Amortized step-wise decay.

Figure: Graphical dynamics.

Conclusion and Open Questions

- We have a dichotomy for approximately counting monotone CNF.
- We also have a partial result in the set packing region (3D matching).

Questions:

- The limit of the correlation decay technique?
- The classification on approximability of monotone #CSP?
- Could we do better on monotone systems with additional local structures e.g. dominating sets?