A Fair Division Solution to the Problem of Redistricting

Z. Landau, O. Reid, I. Yershov

Abstract

Redistricting is the political practice of dividing states into electoral districts of equal population in response to decennial census results to ensure equal representation in the legislative body. Where the boundaries are drawn can dramatically alter the number of districts a given political party can win. As a result, a political party which has control over the legislature, can (and does) manipulate the boundaries to win a larger number of districts, thus affecting the balance of power in the U.S. House of Representatives.

This work introduces a novel solution to the problem of fairly redistricting a state that is motivated by the ideas of fair division. Instead of trying to ensure fairness by restricting the shape of the possible maps or by assigning the power to draw the map to nonbiased entities, this solution ensures fairness by balancing competing interests against each other. Essentially, it is a simple interactive protocol that presents two parties with the opportunity to achieve their fair representation in a state (where the notion of fairness is rigorously defined) and as a result a balanced electoral map is created.

1 Introduction

With the mid-decade redrawing of districts in Texas and recent ballot initiatives for redistricting reform in California and Ohio[1], the subject of political redistricting has received national attention. In the United States, each state is divided into a number of districts proportional to the population of the state. Within each district, an election is held every two years; it is the winners of these elections that comprise the U. S. House of Representatives.

[1] Both of which were defeated.

Every ten years, in response to the national census, the states are redivided into districts to ensure equal representation in the House of Representatives; this redrawing of districts is called redistricting.[2]

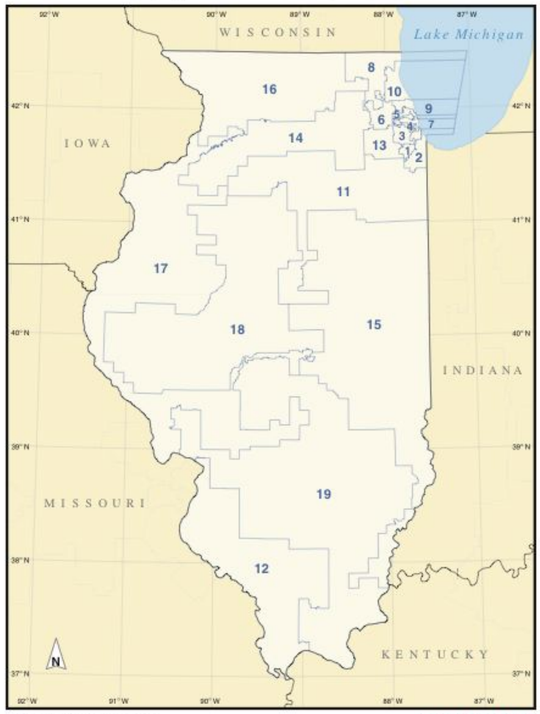
In most states, the responsibility of redrawing the boundaries of districts is assigned to the state legislatures. Where the boundaries are drawn can dramatically alter the number of districts a given political party can win. As a result, a political party which has control over the legislature, can (and does) manipulate the boundaries to win a larger number of districts, thus affecting the balance of power in the U.S. House of Representatives. In theory, with carefully drawn districts, a party that receives $X\%$ of the popular vote can win almost $2X\%$ of the districts, e.g. a party with 55% of the popular vote can win all the districts and a party with 40% of the popular vote can win just under 80% of the districts. This ability of one party to draw districts in such a way as to gain political advantage is viewed as one of the two major problems with redistricting in the United States; we shall refer to this as the problem of *partisan unfairness*. This paper presents a novel solution to the problem of partisan unfairness. (The second problem, which is not a subject of this paper, is the lack of competitive districts.[3])

There is a long history of carefully carving out districts for political gain– this practice, referred to as *gerrymandering*, is named after the Massachusetts governor Elbridge Gerry who in 1812 created a district that supposedly looked like a salamander. Gerrymandering is not merely a historical curiousity; strange shaped districts are commonly found on current district maps. Below are two current maps; notice the shapes of district 17 of Illinois and district 2 of Arizona.

[2] Similar redistricting processes happen in relation to the formation of the state governing bodies.

[3] Most districts are knowingly drawn to give a sizable majority to one party. As a result, even with shifts in public opinion, very few districts have close, competitive elections. To many, this is seen as a negative since those in power need not be very responsive to changing public sentiment.

Illinois Congressional Districts - 19 Districts Total



108th Congress of the United States

USCENSUSBUREAU

The drawing of districts is restricted by legislative and judicial constraints. A careful analysis of these restrictions is given in [**?**]; we give a very brief and incomplete summary here. Each district should contain close to the same number of people. In addition, districts must be drawn taking into account "traditional districting principles": compactness, contiguity, preservation of counties and political subdivisions, preservation of communities of interest and cores of prior districts, protection of incumbents, and compliance with Section 2 of the Voting Rights Act. Section 2 of the Voting Rights Act is designed to protect the racial minority vote from being diluted, mandating that, where appropriate, certain districts have a majority consisting of a racial minority.

Despite these restrictions there remains a lot of freedom as to how a district map can be drawn and partisan unfairness is still a major problem. In the words of Supreme Court Justice Souter in a recent dissent [**?**],". . .the increasing efficiency of partisan redistricting has damaged the democratic process to a degree that our predecessors only began to imagine. "Various proposals to address the partisan unfairness problem have been made over the years. The proposals include transferring the responsibility of drawing the district lines from the legislatures to independent committees, using computer programs to draw the boundaries, limiting the actions of political parties in power by insisting on more rigorously defined geometric criteria like compactness and contiguity for the districts drawn[4] and prohibiting the use of partisan registration and election data.

This work introduces a novel solution to the partisan unfairness problem. Instead of trying to ensure fairness by restricting the shape of the possible maps or by assigning the power to draw the map to nonbiased entities, this solution ensures fairness by balancing competing interests against each other. Essentially, it is an interactive protocol that presents two parties with the opportunity to achieve their fair representation in a state and as a result a balanced electoral map is created.

The protocol has several advantages. First, there is a "good choice feature", which ensures that both parties can always get an almost fair solution for themselves regardless of the actions of the other party, even if their goals are diametrically opposed. Second, the power of deciding on district lines is equally divided between the two parties. Third, if party goals are not diametrically opposed, the protocol provides a fair chance to both parties to achieve a solution that makes them both better off- a feature often found in fair division solutions (for an introduction to fair division, see, for example, [**?**]). Moreover, the protocol is implemented easily: parties' predictions of a voting pattern will guide them in their decisions at every step, and even without a careful analysis reasonable choices may be made. The protocol can also be integrated with judicial constraints; that is, if courts find an adopted redistricting plan not meeting constitutional requirements, a new map can be drawn using the same protocol.

The protocol has a similar feel to a well known solution to the problem of trying to divide a cookie fairly between two parties: one party divides the cookie in two and the other chooses which piece they would like. Both this solution and the solution to the problem of partisan unfairness presented in this paper have the property that each party, by acting in their own interests, can ensure they get their fair share regardless of the actions or preferences of the other. We stress that the "fairness" of the solution presented here is not based on a fixed notion of what is desirable but rather on the preferences of the participants. Just as two people, one who only cares about how many chocolate chips are in their piece and the other who only cares about the

[4]See [**?**] for a description of various mathematical districting models.

size of their piece, can reach an amicable division of the cookie, two parties, with different types of goals for a district map, can achieve a satisfactory solution using the districting protocol.

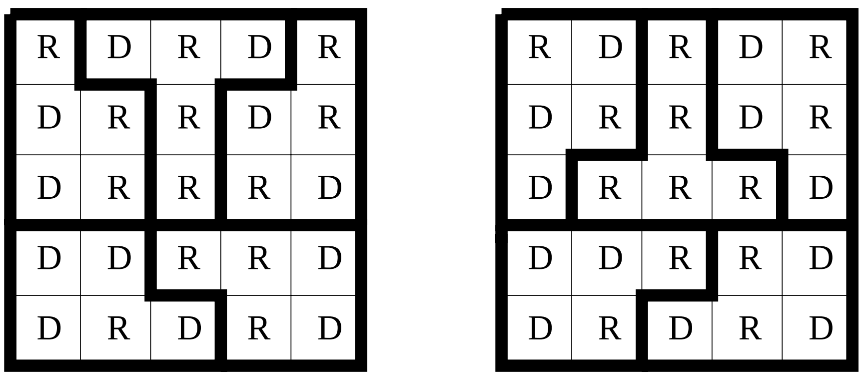
The paper consists of 5 sections. Section 2 introduces notation. Section 3 explains the problem of partisan unfairness, i.e. how under the current districting procedure parties have the ability to use redistricting rules to their advantage. In section 4 we present the new fair division districting protocol that eliminates the problem of partisan unfairness. The analysis of the protocol is conducted in section 5. We begin by describing a good choice property of the protocol (section **??**) and then analyze the protocol in an idealized redistricting scenario with no geometrical constraints (section **??**). We then address potential problems of the protocol (section **??**) and introduce a special augmented fair division redistricting protocol as a solution to one of them. We analyze the effect of the protocol when parties have diametrically opposed preferences (section **??**).

2 Notation

We shall call the land that is to be divided into districts a *state*. A *division* shall refer to a state along with boundaries that divide the state into districts. A *districting protocol* shall be a set of rules for creating a division; those participating in the protocol shall be refered to as parties. The *voting map* shall refer to how each voter will vote in the ensuing election. In general, of course, this is not known precisely, however even partial information about the voting map may be helpful for a non-neutral party involved in a districting protocol.

3 The problem: an inherent unfairness of the current protocol.

In most of the 50 states, as mentioned above, the districting protocol is to have one party draw all the boundaries; we shall call this the *single party districting protocol*. We'll refer to the party with this power as the *drawing* party. If the drawing party's goal is to win as many districts as it can, the strategy is clear: draw boundaries in such a way that each district either a) has a small majority of drawing party voters, or b) has a large majority of the other party's voters. In other words, for any district, the drawing party should strive to either win it by a small margin or lose it by a large margin. Adopting this strategy benefits the drawing party greatly. As an example, consider a 5 by 5 grid that represents 25 voters (a hypothetical state). Suppose we want to divide up the grid into 5 contiguous regions (districts) each containing 5 squares of the grid. The figure below shows two such possible divisions. The letters D and R describe the voting map for the grid: D means a vote for the democratic candidate, R means a vote for the republican candidate. We note that 52% of the voters voted Republican and 48% Democrat. In the first division, Republicans win 80% of the districts while in the second, Democrats win 80% of the districts.



In general, in a single party districting protocol, without geometric constraints a party with $X\%$ of support of the voters, can win just under $\min(2X\%, 100\%)$ of the districts by the strategy of barely winning the districts it wins and badly loosing any district it loses. In reality, the geometric constraints of the layout of the voting map usually mean that this ideal outcome cannot be achieved, however in most cases, the party involved in a single party districting protocol, with even partial knowledge of the voting map can win significantly larger percentage than $X\%$ of the districts. This is not just a theoretical issue, as has been often demonstrated in the United States when the party in control of the districting maps changes. We site two of the most recent examples:

- When Republicans captured control of the Texas legislature in 2002, they redrew the state districts mid-decade, the result was that the Texas delegation changed from 15 Republican and 17 Democratic representatives to 22 Republican and 10 Democratic representatives. [**?**]

- In Michigan, the 2000 election produced 7 Republican representatives and 9 Democratic representatives. After the census, a new district map was drawn resulting in 9 Republican representatives and 6 Democratic representatives in the 2002 election (Michigan lost 1 seat due to the census).[**?**]

It is this inherent unfairness of the current protocol–the ability given to the drawing party in a single party districting protocol to win a dramatically larger fraction of the districts than of the constituent voters– that the districting solution proposed in this paper avoids. In contrast, as we shall see, the protocol proposed here ensures that either party, with knowledge of the voting map, can ensure that their party wins a percentage of districts that is very close to the percentage of support they have from the voters.

4 The protocol

We introduce a new protocol for determining the division of a state. It will involve three parties: two parties with vested interest in the division (e.g. the democratic and republican parties, or the majority and minority party in a state) called parties A and B, and an independent third party, party I.

Let us suppose that we would like to produce a division of a state into n districts with d people in each district. Before describing the protocol, we need to define the notion of a *k split*. Consider splitting the state into two contiguous pieces, call them X and Y, such that the size of the population in the piece X is kd; we will call the pair (X,Y) a *k split* of the state.

We can now describe the steps of the protocol:

Fair division districting protocol

1. For each i, $1 \leq i \leq n-1$, party I constructs an i split (X_i, Y_i) such that
$$X_1 \subset X_2 \subset \cdots \subset X_{n-1}.$$
(Here $X_1 \subset X_2$ means region X_2 contains region X_1, etc.)

2. For each i, both parties are asked which they would prefer:
   (a) a division created by allowing party A to divide X_i into i districts and party B to divide Y_i into $n-i$ districts.

(b) a division created by allowing party A to divide Y_i into $n - i$ districts and party B to divide X_i into i districts.

Notice that neither party will choose to divide X_1 or Y_{n-1} as these two regions are the size of a single district and therefore no further division would be done to them. Thus for $i = 1$, party A will choose option (b) while party B will choose option (a). Similarly, for $i = n - 1$, party A will choose option (a) and party B will choose option (b).

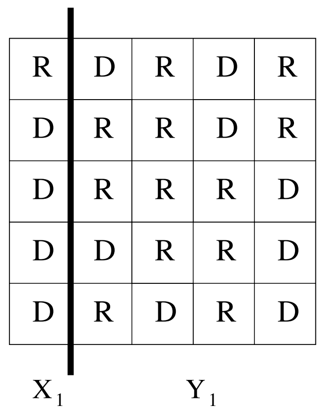
3. Suppose there exists an i such that parties A and B both prefer the same option in the choice above. Then create a division using that option.

4. If no such i exists, this means that the parties have opposite preferences for each i. Randomly choose an i_0, $1 \leq i_0 \leq n - 2$ for which party A prefers option (b) for $i = i_0$ and switches preferences to option (a) when $i = i_0 + 1$. (This scenario is guaranteed to occur at least once since party A prefers option (b) when $i = 1$ and prefers option (a) when $i = n - 1$.) Randomly choose to divide the state from the following four options:

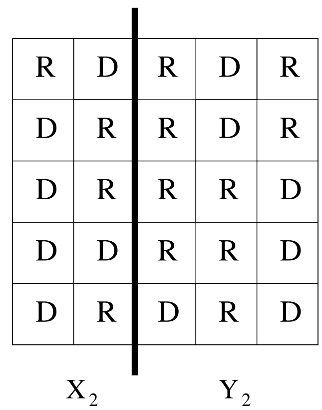
   i. option (a) for $i = i_0$,
   ii. option (b) for $i = i_0$,
   iii. option (a) for $i = i_0 + 1$,
   iv. option (b) for $i = i_0 + 1$.

4.1 Application of the protocol to the 5x5 example.

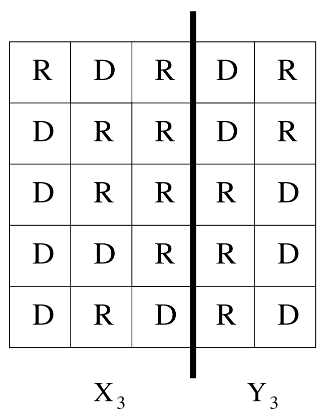
For clarity, we will apply the protocol to produce a division of the voting map given in section **??**. We shall assume that both parties are solely concerned with winning as many districts as they can. Let us suppose that the k-splits consist of vertical lines:



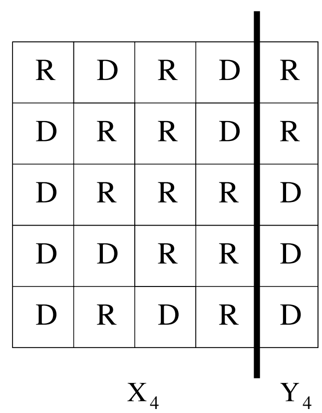
1−Split



2−Split



3−Split



4−Split

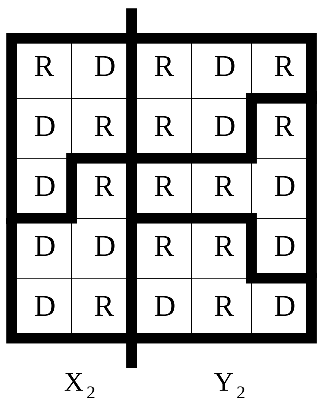
A brief analysis shows that Democrats and Republicans would both prefer to divide Y_1, Y_2, X_3, X_4. Thus we find ourselves in step 4 of the protocol. We have $i_0 = 2$ and thus we randomly pick between the following four options (possible maps that achieve these results appear below):

i. Democrats dividing X_2 and Republicans dividing Y_2; with the outcome being Democrats winning 2 districts and Republicans winning 3 districts.

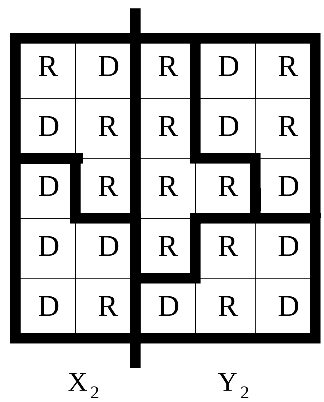
ii. Democrats dividing Y_2 and Republicans dividing X_2; with the outcome being Democrats winning 3 districts and Republicans winning 2 districts.

iii. Democrats dividing X_3 and Republicans dividing Y_3; with the outcome being Democrats winning 3 districts and Republicans winning 2 districts.

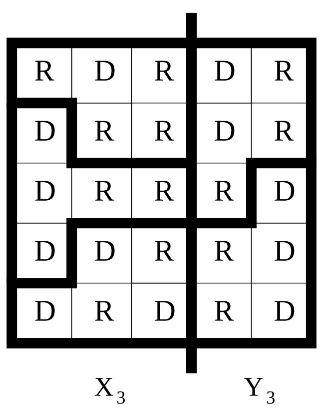
iv. Democrats dividing Y_3 and Republicans dividing X_3; with the outcome being Democrats winning 2 districts and Republicans winning 3 districts.



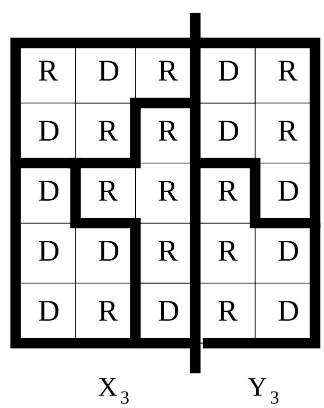
i. D: 2, R: 3



ii. D: 3, R: 2



iii. D: 3, R: 2



iv. D: 2, R: 3

Since there are 5 districts to be divided amongst the two parties, the two outcomes closest to the 48% Democratic, 52% Republican split of the popular votes is each party winning 2 or 3 districts. Notice that, as expected, all four choices lead to such a division.

5 Analysis of the fair division districting protocol.

5.1 The good choice property.

We analyze this protocol from party A's perspective. We shall assume that party A has made a model of the voting map and wants to get the best possible outcome for that voting map. Recall that for a given i, if option (a) is chosen

i. party A divides X_i,

ii. party B divides Y_i.

Whereas if option (b) is chosen,

iii. party A divides Y_i,

iv. party B divides X_i.

Suppose party A's goal is to win as many districts as possible. Notice that the results of actions i. and iii. together consist of the best possible division (of the whole state) for party A conditioned on the requirement that the division includes the boundary that splits X_i from Y_i. Similarly, the results of actions ii. and iv. together consist of the best possible division for party B conditioned on the requirement that the division includes the boundary that splits X_i and Y_i. Therefore the average, over the two options, of the number of districts won by party A is equal to the average of the best scenario for party A and another scenario which might be the worst scenario for party A (if party B's goal is also to win as many districts as possible). Thus at least one of the two options (a) or (b) allows party A to win at least the average of the most and the least districts party A can win.

In fact, this same argument holds for goals other than that of winning the most districts possible. A party could rate each district in a division (assigning it a number) and then add these numbers up for all the districts to give a rating for the division. The previous case of maximizing the number of districts won can be viewed as an example of this: the party would assign a 1 to districts it expects to win and a 0 to those it expects to lose. Thus the rating assigned to a division is exactly the number of districts the party expects to win. The more general rating system (of allowing the ratings of a district to be any number) allows a party to take other considerations into account. Politically, these considerations can be important, a few examples of such considerations include:

- perhaps some district has an incumbent who is on an important congressional committee and so winning that district is more valuable to the party (and thus, if winnable, it would be rated higher than other winnable districts),
- perhaps some district has an important landmark in it (a stadium or a construction project) and would be worth more to a party than some other district,
- perhaps some district encompasses the supporters of two incumbents from the opposition party, thus even though the district will be lost, the elimination of one strong incumbent from the other party is valuable.

We will call this kind of rating process an *additive rating system*, i.e. any system that rates a division by adding up the ratings of the individual districts. Define the *average rating for an* (X, Y) *split* to be the average of the highest and lowest ratings among all divisions that include the boundary between A and B.

The previous discussion for the goal of winning as many districts as possible naturally extends to say:

The good choice property:

> *For any additive rating system, at least one of the options (a) or (b) for the i-split* (X_i, Y_i) *will create a division with rating at least as big as the average rating for the* (X_i, Y_i) *split.*

The good choice property above is the core reason why the fair division redistricting protocol is a good solution to the partisan unfairness problem. The property says that either party, no matter what additive rating system they prefer, will always be presented with an average or better than average choice.

5.2 An example: redistricting with no geometric constraints.

Consider the scenario where there are no geometric restrictions on how the divisions can be made. Recall from section **??** that in this scenario $X\%$ of the popular vote for the drawing party could theoretically turn into the drawing party winning almost $\min(2X\%, 100\%)$ of the districts. If we let x represent the fraction of the popular vote received by party A, then party A wins about a fraction $b_A \approx \min(2x, 1)$ of the districts in the best division, and a fraction $w_A \approx 1 - \min(2(1-x), 1) = \max(2x-1, 0)$ of the districts in the worst division (i.e. one minus the best division for party B). Notice that for $x \geq .5$, $b_A \approx 1$, $w_A \approx 2x - 1$ and for $x < .5$, $b_A \approx 2x$, $w_A \approx 0$. Thus in either case the average of the best and worst outcomes, is party A winning a fraction

$$\frac{b_A + w_A}{2} \approx x$$

of the districts. We apply this observation to the fair division districting protocol. Suppose that party A receives a fraction x of the popular vote. Let x_1 and x_2 be the fraction of the popular vote received in X_i and Y_i respectively. Thus

$$\frac{i}{n}x_1 + \frac{n-i}{n}x_2 = x.$$

Thus the average number of districts won by party A in scenarios i and iv above is approximately the fraction x_1 of the i districts. Similarly, the average number of districts won by party A in scenarios ii and iii is approximately the fraction x_2 of the $n - i$ districts. So combined, the average of options (a) and (b), i.e. the average rating for the (X_i, Y_i) split, is approximately

$$x_1 i + x_2(n-i) = (x_1\frac{i}{n} + x_2\frac{n-i}{n})n = xn$$

. We see therefore that one of the two choices (a) or (b) can be chosen to ensure that party A can win at least an approximate x fraction of districts, i.e. they can ensure winning approximately the same fraction of districts as the fraction of votes they received in the entire state.

5.3 Two potential problems.

We now turn to address two potential concerns about the protocol:

- *The protocol seems to favor the minority party.* At first glance, this process may appear to unfairly favor the minority party since it seems to average the two parties desires equally, regardless of how small a minority party is. In fact this is not the case, for as a minority party becomes weaker, both its best and worst divisions achieve less and less of what the minority party desires. In other words, the mere power to draw districts is useless if you don't have voters that support you. A careful reading of the example in section **??** should convince the reader that the protocol shares the power of drawing districts in a desirable way.

- *The placement of the* (X_i, Y_i) *split can dramatically effect the results.* Notice that given an additive rating system, the average rating for an (X, Y) split may differ from the average of the best and worst rating over all divisions. In other words, insisting that the division includes the boundary between X and Y may unduly favor one party. This, indeed, is possible but two observations suggest that this effect will not be dramatic. First, the choice of split is made by an independent (neutral) third party, and therefore is in some sense random. Second, as we have seen in section **??**, in the case where there are no geometric constraints, this split has virtually no effect for the goal of winning as many districts as possible; the average ratio for an (X, Y) split is very close to the average of best and worst ratings amongst all divisions.

  Nevertheless, the possibility of getting a "bad" split (for some party) still exists. This is one of the reasons that we introduce the augmented protocol in the next section which essentially does the fair division redistricting protocol multiple times and chooses a division that both parties like.

5.4 The augmented protocol.

Augmented fair division redistricting protocol

1. Apply most of the fair division protocol N times: for each application, return a description of how a division will be created- i.e. a split (X, Y) and the option ((a) or (b)) for how the division will be made.

2. Ask each party to rank in order of preference the N proposed divisions. Each proposed division then has two rankings. Select the proposed division whose worse ranking is best. If there are 2 such proposals (there can be at most 2), randomly choose one.

3. Create a division based on this proposed division.

Notice that whatever proposed division is picked, it will be in the top $\frac{N}{2}+1$ on both parties lists. (Since more than half the proposed divisions are ranked in the top $\frac{N}{2}+1$.)

It is reasonable to assume that most splits will not particularly favor either party. As mentioned above, this augmented protocol ensures that a rare "bad" split for a particular party will not come into play (since the effected party would put such a split towards the bottom of their rankings).

We see therefore, that the good choice property, when coupled with the augmented protocol, implies that a party should be satisfied if the division is created by an option that was chosen by that party. Thus if the division occurs as a result of step 3, both parties should be satisfied.

We are left to analyze what happens when the chosen division occurs as a result of step 4 of the protocol. We show in the next section that the amount of dissatisfaction of one party should be small.

5.5 Analysis of the effect of step 4.

If, for none of the i-splits, the two parties prefer the same option for dividing the district then we proceed to step 4 of the protocol. Let us suppose the random choice in step 4 is i., that is, option (a) for $i = i_0$. The analysis is very similar for all the other cases. In this case, party B preferred this option so party B should be satisfied with the division as discussed in the previous section. Party A, however, prefered option (b), to divide Y_{i_0} and have party A divide X_{i_0}. Notice, however, that party A would prefer to divide up X_{i_0+1}, and X_{i_0+1} only differs from X_{i_0} by a small region with a population equal to the size of a single district. (Similarly the complementary pieces Y_{i_0} and Y_{i_0+1} only differ by this same small region). Because party A prefers option (b) for the (X_{i_0}, Y_{i_0}) split and option (a) for the (X_{i_0+1}, Y_{i_0+1}) split (and because X_{i_0+1} and X_{i_0} do not differ by very much), it is reasonable to expect that party A's preference for option (b) over option (a) for the (X_{i_0}, Y_{i_0}) split is mild. If indeed this is the case, then party A's discontent with the division would only be mild as we have shown (by the good choice property) that party A would be satisfied with the slightly better option of (b).

Even though X_{i_0} and X_{i_0+1} differ by a small amount, one can construct scenarios where that small amount makes a big difference. However, recall that the creation of X_{i_0} and X_{i_0+1} was done by an independent party and therefore one would expect this type of scenario to be rare. Again, choosing to use the augmented protocol would ensure that this rare scenario would not come into play.

References

[BT] S. Brams, A. Taylor, Fair division. From cake-cutting to dispute resolution. Cambridge University Press, Cambridge, 1996.

[CMPRS] Evaluation and Optimization of Electoral Systems, P. Grilli di Cortona, C. Manzi, A. Pennisi, F. Ricca, and B. Simeone, *SIAM Monographs on Discrete Mathematics and Applications.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[OC] Office of the Clerk, U.S. House of Representatives website: `http://clerk.house.gov/members/electionInfo/elections.html`

[R2000] Redistricting Law 2000. Denver, CO: National Conference of State Legislatures, 1999. Available at `http://www.senate.leg.state.mn.us/departments/scr/redist/red2000/red-tc.htm`

[Vie] Dissent of Justice Souter, *VIETH V. JUBELIRER (02-1580) 541 U.S. 267* (2004)