Optimal parallel repetition for projection games on low threshold rank graphs

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Two-prover one-round games

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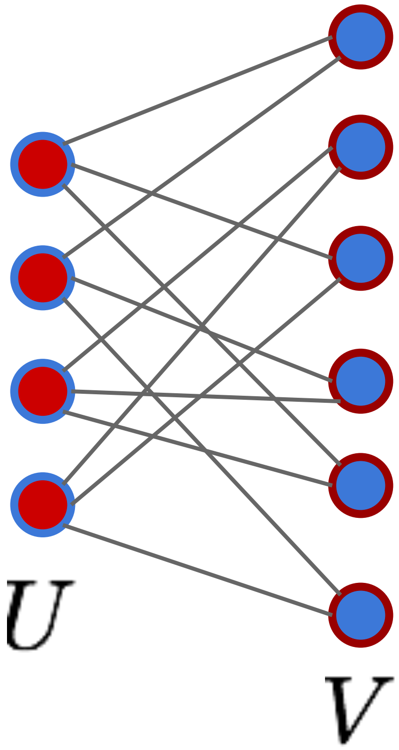

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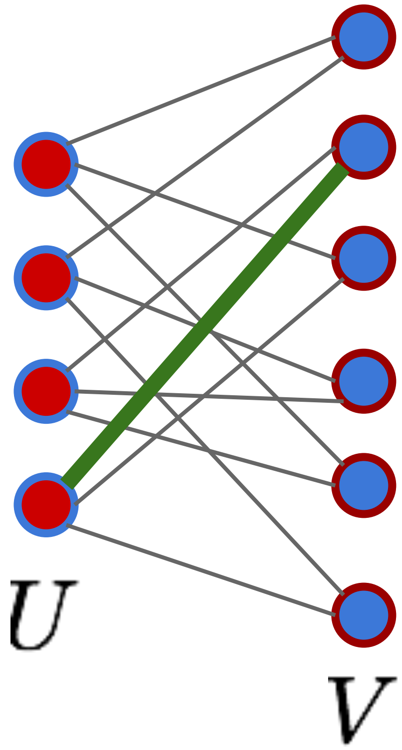

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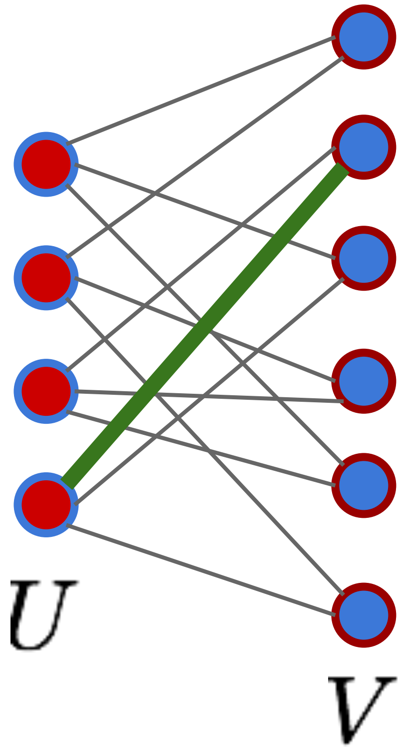

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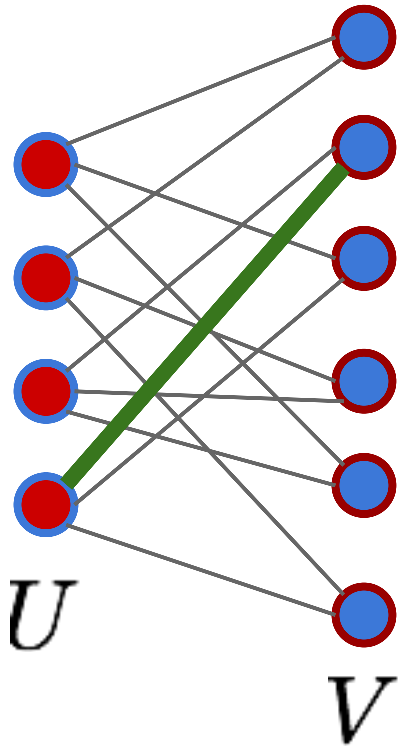

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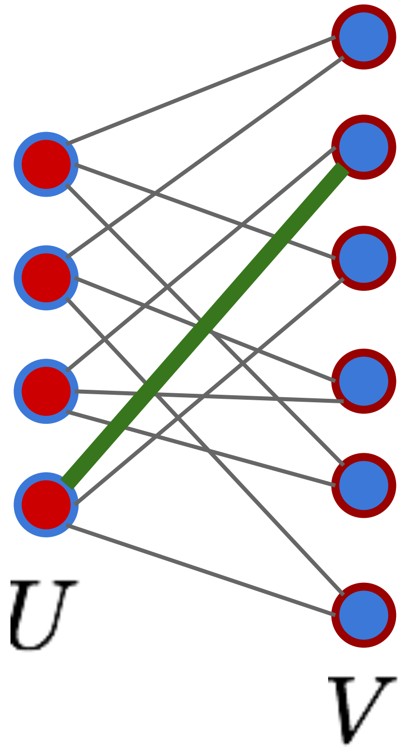

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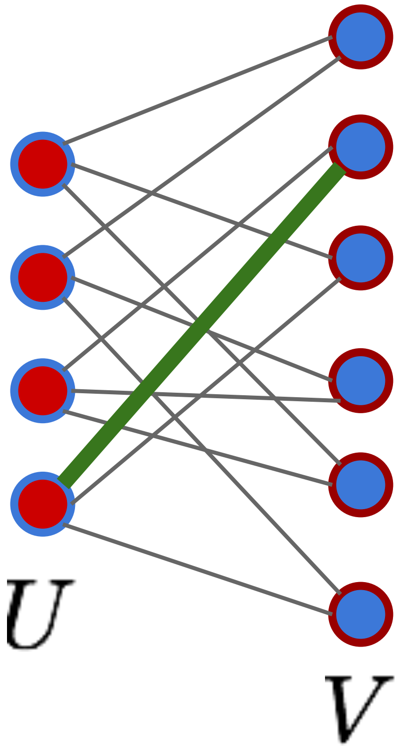

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(biregular)

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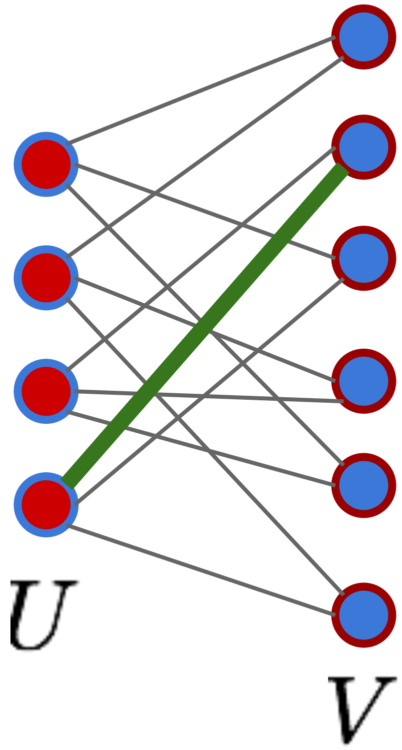

Projection games

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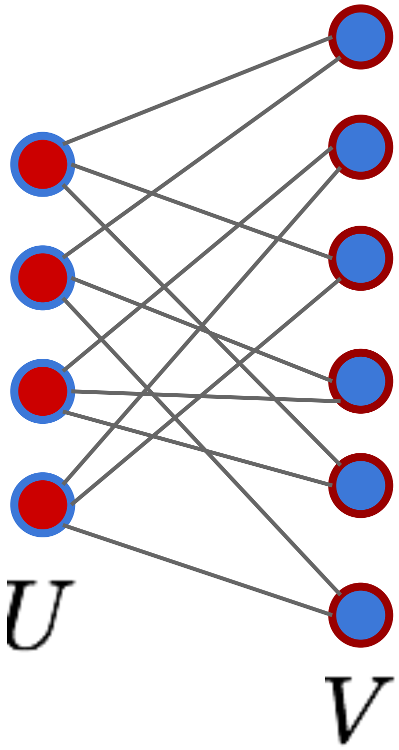

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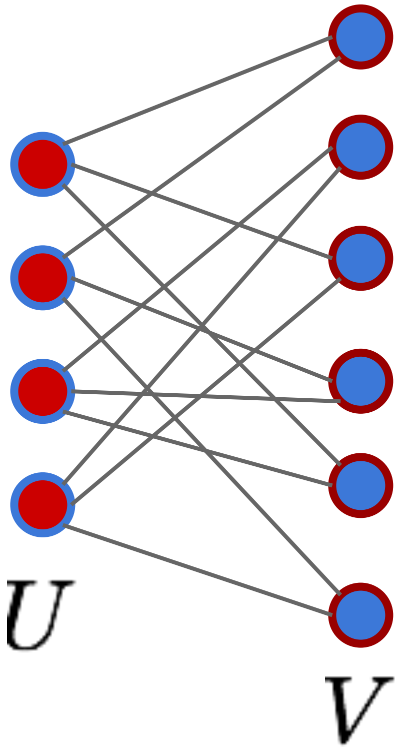

Projection games

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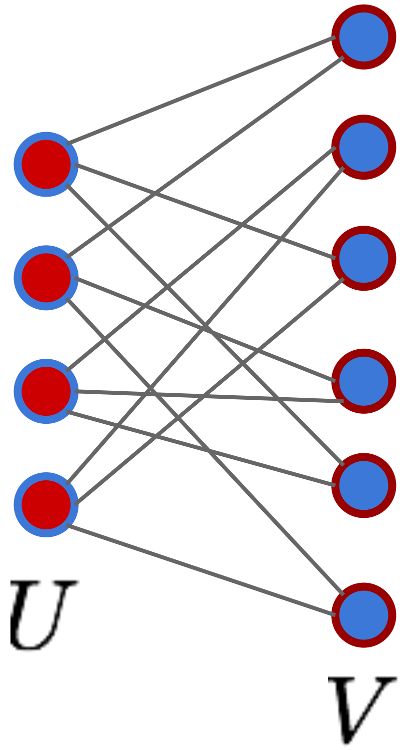
G is a **projection game** if for every *b*, only exists one *a* to satisfy π_{uv}



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$\boxed{G^{\otimes k} \text{ is the parallel rep. of } G \;\; k \text{ times with itself}}$

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no strong parallel repetition in general

Parallel repetition theorems

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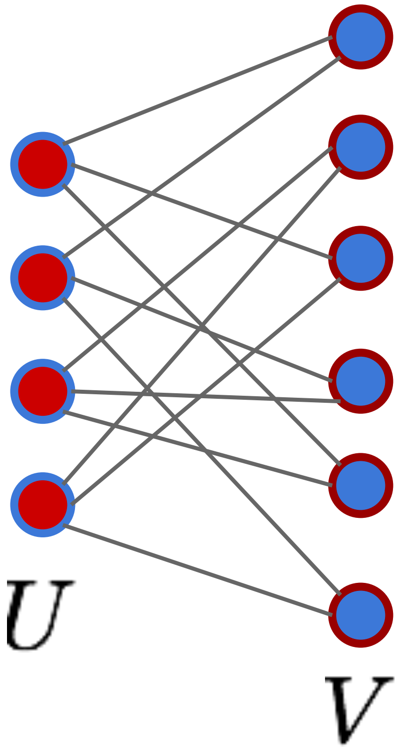
for projection games

Expanding games

Bipartite graph $G = (U \cup V, E)$



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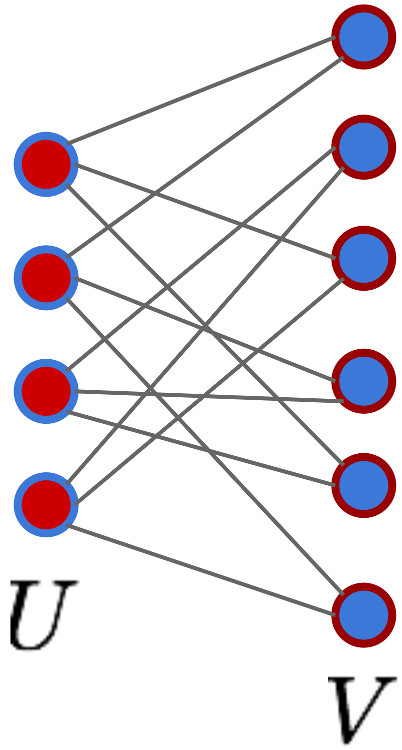
G's spectrum via the SVD:

$$1 = \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq 0$$



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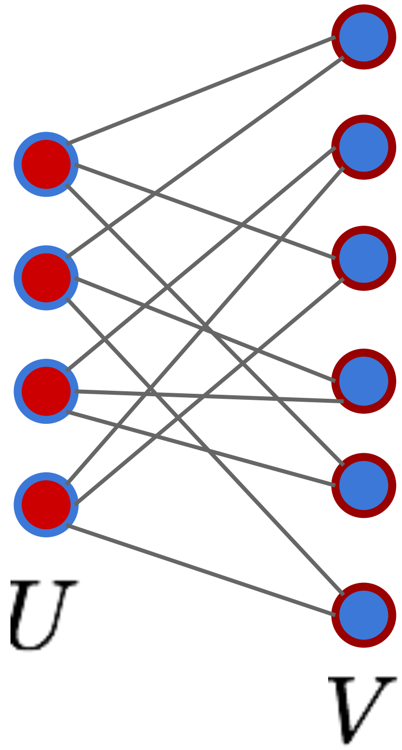

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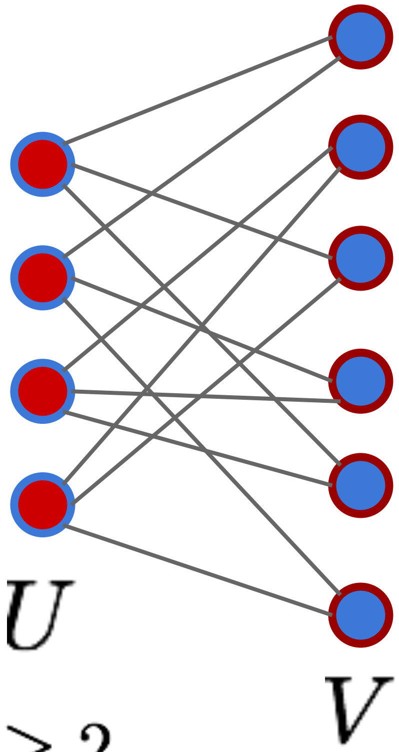

G is an ***expanding game***

if σ_2 is small.

G has ***low threshold rank***

if σ_k is small for some small $k \geq 2$.

Strong PR for Expanding Games

Let G be a projection game with 2nd largest singular value σ_2. If $\mathrm{val}(G) = 1 - \eta$, then

$$\mathrm{val}(G^{\otimes n}) \leq (1 - (1 - \sigma_2^2)^2 \cdot \eta)^{\Omega(n)}$$

[Raz and Rosen]

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Q1: Does this have a **tight** depencence on σ_2 ?

Q2: What about games with **low threshold rank**?

Main result

Let G be a projection game with *k*-th largest singular value σ_k. If $\mathrm{val}(G) = 1 - \eta$, then

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Improves on Raz and Rosen when *k* = 2.

Optimal for all fixed *k*.

Proof strategy

Parallel repetition framework of **[Dinur Steurer]**

+

Strong Cheeger's inequality due to **[KLLGT13]**

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[Dinur Steurer 2014]

New framework for proving parallel repetition theorems using **linear algebra**.

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New proof that $\mathrm{val}(G^{\otimes k}) = (1-\eta^2)^{\Omega(k)}$ if G is a **projection game** [originally from Rao]

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Connects PR to **Cheeger's inequality**.

Cheeger’s inequality

Technique for finding **non-expanding** sets in graphs.

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Given a graph $G = (V, E)$, let λ_2 be the second-smallest eigenvalue of its Laplacian.
Then $$\frac{1}{2}\lambda_2 \leq \phi(G) \leq \sqrt{2\lambda_2}$$

Cheeger's inequality

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(**conductance** of G)

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(loses a square)

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[Dinur Steurer 2014]

Proof that $\mathrm{val}(G^{\otimes k}) = (1 - \eta^2)^{\Omega(k)}$ if G is a **projection game** uses Cheeger's inequality.

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Better Cheeger's inequality ⇒ Better PR

[Dinur Steurer 2014]

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Strong Cheeger's inequality ⇒ **Strong** PR

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$$\phi(G) \leq O(k) \frac{\lambda_2}{\sqrt{\lambda_k}}.$$

[KLLGT13]

A **strong** Cheeger's inequality for graphs with **low threshold rank**

($\approx \lambda_2$ when λ_k is constant. No square lost!)

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Main result

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$$\mathrm{val}(G^{\otimes n}) \le \left(1 - \frac{\sqrt{1-\sigma_k^2}}{k} \cdot \eta\right)^{\Omega(n)}.$$

Second result

Let G be a projection game with 2nd largest singular value σ_2. If $\mathrm{val}(G) = 1 - \eta$, then

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Second result

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Improves on [Raz and Rosen]

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Improves on **[Raz and Rosen]**

Simple proof: **no Cheeger's needed**

Thanks!