Beating the random assignment for constraint satisfaction problems of bounded degree

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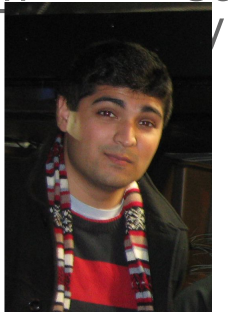

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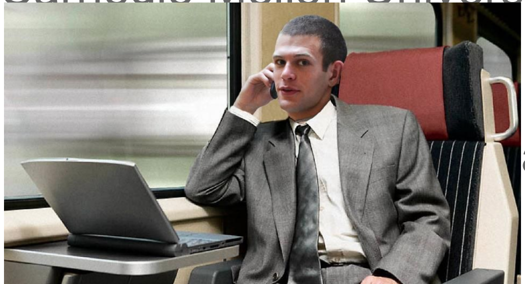

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3XOR

$$x_1 x_3 x_5 = 1$$
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The end.

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This work: What if instance is **degree bounded**, i.e. every variable appears in $\leq$ **D** clauses?

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[Hås00]: Can approximate any degree bounded CSP to factor $\mu + O(1/\mathbf{D})$.

[Tre01]: For 3XOR, can't approximate better than $\frac{1}{2} + O(\mathbf{D}^{-1/2})$.

Our inspiration

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Shtetl-Optimized

The Blog of Scott Aaronson

Quantum computing news items (by reader request)

Main theorem

Given a **D**-degree bounded **k**-Lin instance, can find (in poly-time) an assignment **x** such that

$$|\mathbf{val}(\mathbf{x}) - \tfrac{1}{2}| \geq \Omega_{\mathbf{k}}(\mathbf{D}^{-1/2}).$$

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If **k** is **odd**, then for **x** or its negation -**x**,

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(and this is easily shown to be tight)

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After our work...

[FGG]: for 3Lin, the **QAOA** algo finds **x** s.t.

$$\mathbf{val}(\mathbf{x}) \geq \tfrac{1}{2} + \Omega(\mathbf{D}^{-1/2} \log(\mathbf{D})^{-1}).$$

Let’s do the proof for $k = 3$.

Step 1: Decoupling

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[KN08]: can assume WOLOG that instance is decoupled

Step 2: The algorithm

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$$\Rightarrow \text{want } \sum_i b_i\, \mathbf{y}_{i1}\mathbf{z}_{i2}\mathbf{z}_{i3} \geq \mathbf{mD}^{-1/2}$$

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Generalizing to general *k*-XOR

Can generalize this approach.

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Or: **[DFKO07]**: Given a low degree poly p(**x**) where all variables are low-influence, there exists an **x** s.t. |p(**x**)| is large.

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(if you work out parameters, actually immediately yields existential version of our main result!)

Algorithmic DFKO

Thm: Let $g:\{-1,1\}^n \to \mathbf{R}$ have degree $\mathbf{k}$ and variance $\mathbf{1}$. Suppose $\mathrm{Inf}_i[g] \le O_{\mathbf{k}}(\mathbf{t}^{-2})$ for all i. Then can find an $\mathbf{x}$ s.t.

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in poly-time.

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Pf: Line-by-line **[DFKO]**, note that it constructivizes.

Generalizing to all CSPs?

Suppose C_1,..., C_m are constraints.

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C_i's are XORs: Fourier expansions don't overlap, so **influence** = **degree** of var.

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C_i's are **not** XORs: Fourier expansions may overlap, so can get weird cancellation

Counterexample

Maj(**x**,**y**,**z**)

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-**x**

-**y**

-**z**

XOR(**x**,**y**,**z**)

Counterexample

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- **every** assignments gets ½

Counterexample

Maj(x,y,z)

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-y

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XOR(x,y,z)

Maj(x,y,z) = ½ ($x + y + z - xyz$)

- 3-degree bounded
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Counterexample

$\mathrm{Maj}(x,y,z)$

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$-x$

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$\mathrm{XOR}(x,y,z)$

$\mathrm{Maj}(x,y,z) = \frac{1}{2}(x + y + z - xyz)$

- 3-degree bounded
- random gets ½
- **every** assignments gets ½
- so can't beat random

Generalizing to all CSPs?

In summary: no CSPs beyond XORs immediately comes to mind. So this is a theorem about XORs.

What about that weird odd/even stuff?

Recall: for odd **k**, **val**(**x**) $\geq \frac{1}{2} + \Omega_{\mathbf{k}}(\mathbf{D}^{-1/2})$.

For even **k**, |**val**(**x**) - ½| $\geq \Omega_{\mathbf{k}}(\mathbf{D}^{-1/2})$.

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Max-Cut on **n**-vertex complete graph:

What about that weird odd/even stuff?

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For even **k**, |**val**(**x**) - ½| ≥ $\Omega_{\mathbf{k}}(\mathbf{D}^{-1/2})$.

Max-Cut on **n**-vertex complete graph:

- **n**-degree-bounded
- **OPT** = ½ + O(1/**n**).

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Def: the instance is **triangle-free** if this graph is triangle-free

Some triangle-free algos

[FGG15]: given a **D**-degree-bounded **triangle-free** 3Lin instance, **QAOA** finds **x** s.t.

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[Us]: the same guarantee for **any** CSP

Open directions

Characterize CSPs for which

- you can get a **$D^{-1/2}$** advantage
- you can get **$D^{-1/2}$**-far from the mean (even if in the wrong direction)

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Prove that **QAOA** gets ½ + $\Omega($**D**$^{-1/2})$.


Thanks!