Stabbing Delaunay Tetrahedralizations

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Abstract

A Delaunay tetrahedralization of n vertices is exhibited for which a straight line can pass through the interiors of $\Theta(n^2)$ tetrahedra. This solves an open problem of Nina Amenta, who asked whether a line can stab more than $O(n)$ tetrahedra. The construction generalizes to higher dimensions: in d dimensions, a line can stab the interiors of $\Theta(n^{\lceil d/2 \rceil})$ Delaunay d-simplices. The relationship between a Delaunay triangulation and a convex polytope yields another result: a two-dimensional slice of a d-dimensional n-vertex polytope can have $\Theta(n^{\lfloor d/2 \rfloor})$ facets. This last result was first demonstrated by Amenta and Ziegler, but the construction given here is simpler and more intuitive.

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Can a straight line intersecting a Delaunay tetrahedralization of n vertices pass through the interiors of more than $O(n)$ tetrahedra? This question has been posed as an open problem by Nina Amenta.

Figure 1 resolves the question by illustrating that a straight line can stab the interior of every tetrahedron of a quadratic-size tetrahedralization. First imagine a slightly different tetrahedralization in which the vertices lie on two lines: the z-axis (vertical), and the line $y = 1$, $z = 0$, which is parallel to the x-axis. Distribute $n/2$ vertices along each line. The Delaunay tetrahedralization of these vertices—the *only* tetrahedralization of these vertices—has $(n/2 - 1)^2$ tetrahedra, because each edge on the z-axis is paired with each edge on the horizontal line to form a tetrahedron. (See Figure 2, left.)

Perturb the x-coordinate of each vertex on the z-axis so that the sequence of vertical edges forms a zigzag, as illustrated in the left half of Figure 1. If the perturbation is small enough, all of the tetrahedra in the original tetrahedralization also appear (in perturbed form) in the Delaunay tetrahedralization of the perturbed vertices, along with some new tetrahedra that fill in cavities on the perturbed sides of the tetrahedralization. (See Figure 1, right.)

The z-axis intersects the boundary of every tetrahedron in this Delaunay tetrahedralization, because it intersects every edge of the zigzag. The dashed line $x = 0$, $y = \epsilon$ stabs the interior of every tetrahedron if ϵ is chosen to be sufficiently small.

To verify that each edge of the zigzag does in fact belong to at least $n/2 - 1$ Delaunay tetrahedra, see Figure 2. For brevity, call $y = 0$ the *blue plane* and $y = 1$ the *green plane*. Every vertex of the construction lies on one of the two planes. For any sphere, call its intersection with the blue plane its *blue circle* and its intersection with the green plane its *green circle*. (See Figure 2, left.) If a sphere passes through four affinely independent vertices, and no other vertex lies on or inside its blue circle or its green circle, the convex hull of the four vertices is (by definition) a Delaunay tetrahedron. Observe that any sphere that intersects both planes can be identified by four independent parameters: the x- and z-coordinates of its center, and the radii of its blue and green circles.

Consider the two-dimensional Voronoi diagram of the zigzag vertices, where the Voronoi diagram is defined in the blue plane (Figure 2, center). Call the edges of this Voronoi diagram the *blue edges*. Likewise, the *green edges* are the edges of the Voronoi diagram of the other vertices, defined in the green plane. If the two diagrams are overlaid by projection onto the x–z plane (Figure 2, right), each intersection of a blue edge and a green edge represents a Delaunay tetrahedron whose circumscribing sphere is centered at the x-

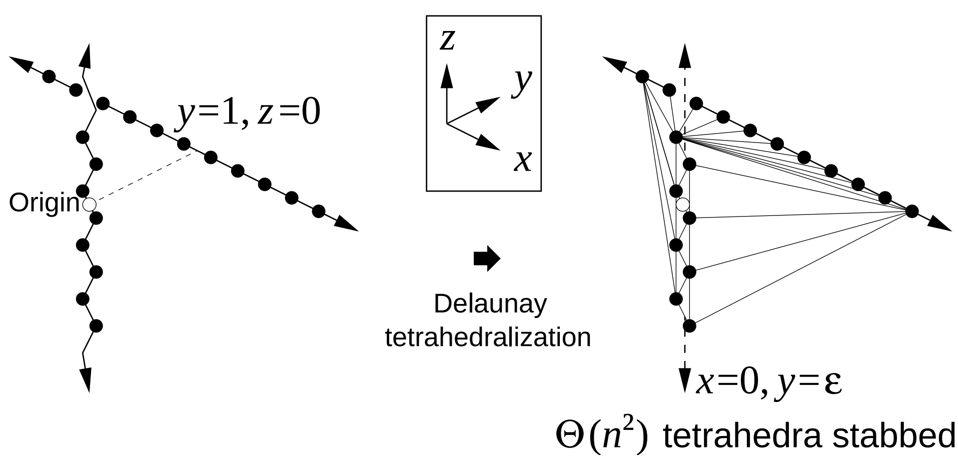

Figure 1: A Delaunay tetrahedralization with more than $(n/2 - 1)^2$ tetrahedra, for which a single line stabs the interior of every tetrahedron.

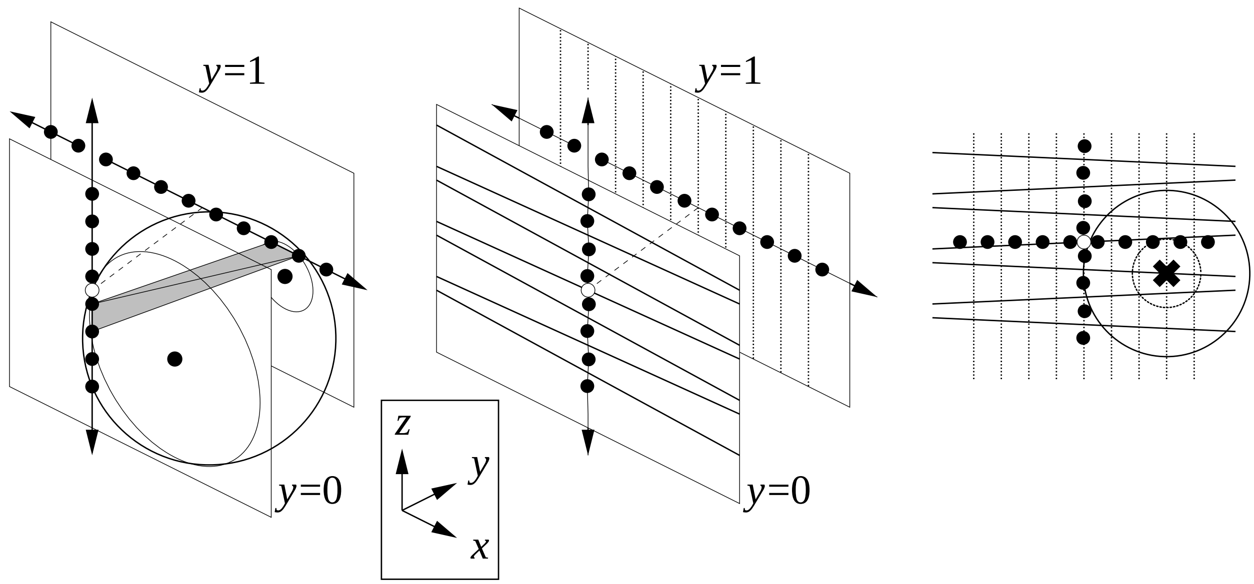

Figure 2: Left: each Delaunay tetrahedron has a circumscribing sphere whose intersections with the planes $y = 0$ and $y = 1$ are empty circles. Center: two two-dimensional Voronoi diagrams, defined within the planes $y = 0$ and $y = 1$. Right: intersections of the overlaid Voronoi diagrams indicate centers of circumscribing spheres of Delaunay tetrahedra.

and z-coordinates of the intersection point. This follows because the radii of the blue and green circles can be chosen independently to yield a sphere that intersects two green vertices and two blue vertices, and does not enclose any vertex.

As long as the zigzag edges do not tilt too far, each blue Voronoi edge that is dual to a zigzag edge intersects (in the overlay) every one of the $n/2 - 1$ green edges. It follows that each zigzag edge is included in at least $n/2 - 1$ Delaunay tetrahedra. How far is "not too far"? Each zigzag is bisected by a blue edge, and the intersection of two blue bisectors cannot fall between two green edges.

The construction is easily adapted if we require that the vertices be in general position (that is, no four vertices are coplanar). If we perturb each vertex of the construction in Figure 1 by a sufficiently small random vector (a different vector for each vertex), all of the tetrahedra in the original tetrahedralization remain in the Delaunay tetrahedralization of the twice-perturbed vertices, and are still stabbed by the dashed line. Even more Delaunay tetrahedra appear; not all of these are stabbed by the dashed line, but the total number stabbed is quadratic.

The construction adapts to higher dimensions. In five-dimensional space, for instance, add a third line that is parallel to the v-axis and is offset from the origin by one unit in the w-direction. Distribute $n/3$ vertices along each of the three lines. The zigzag is now formed by perturbing the v- and x-coordinates of each vertex on the z-axis. The v-coordinate of each zigzag vertex is equal to its x-coordinate, so every edge of the zigzag passes through the z-axis, just as in the three-dimensional example. The five-dimensional Delaunay triangulation of these vertices has $(n/3 - 1)^3$ 5-simplices before the perturbation is applied (creating the zigzag), and more after. Each pre-perturbation 5-simplex is defined by choosing one edge from each of the three lines and taking their convex hull. The line $v = 0$, $w = \epsilon$, $x = 0$, $y = \epsilon$ stabs the interior of every 5-simplex of the post-perturbation triangulation.

For the five-dimensional construction, the blue and green planes are replaced by three parallel 3-spaces (each of them orthogonal to the w- and y-axes); the roles of the blue and green edges are played by Voronoi

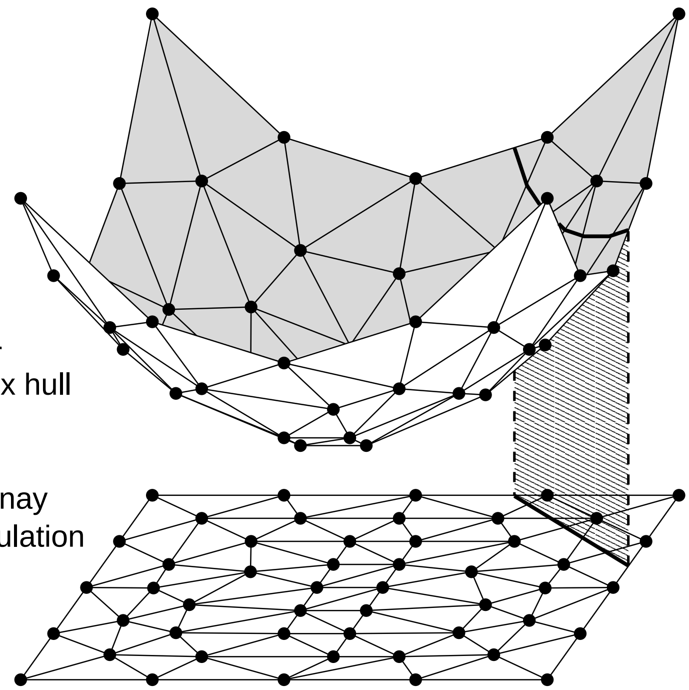

Figure 3: The parabolic lifting map demonstrates that a stabbing line through a Delaunay triangulation is combinatorially related to a two-dimensional cross section of a convex hull.

2-faces in three colors; and each point of the overlay where three differently colored 2-faces intersect represents the center of the circumscribing 4-sphere of a Delaunay 5-simplex. There are exactly $(n/3-1)^3$ of these intersections.

For each additional two dimensions, the number of d-simplices that can be stabbed increases by a factor of $\Theta(n)$. If d is fixed, the maximum stabbing number matches the $\Theta(n^{\lceil d/2 \rceil})$ upper bound on the number of simplices in a d-dimensional triangulation. If growth in both n and d is accounted for, the construction given here stabs $\Theta((n/\lceil d/2 \rceil)^{\lceil d/2 \rceil})$ d-simplices. There is an asymptotic gap between this stabbing number and the upper bound of $\Theta(d^{-1/2}(en/\lceil d/2 \rceil)^{\lceil d/2 \rceil})$ provided by McMullen's upper bound theorem for polytopes [4]. (Here e is the base of the natural logarithm, and the latter expression depends on the assumption that $d \in o(\sqrt{n})$).

A well-known transformation called the *lifting map* of Seidel [5, 3] maps each vertex onto a paraboloid in a space one dimension greater, so that the Delaunay triangulation of the vertices is a projection of the lower convex hull of the mapped vertices. For example, $(x, y, z) \to (x, y, z, x^2 + y^2 + z^2)$ maps the vertices of a Delaunay tetrahedralization to a paraboloid in E^4. As Figure 3 illustrates, each d-simplex of the Delaunay triangulation corresponds to a facet of the lower surface of the convex hull of the lifted vertices. A stabbing line through the triangulation corresponds to a planar cross section of the lower convex hull; the vertical plane in the figure is an example of a cross-sectional plane.

Hence, the stabbing result implies that a two-dimensional slice of a d-dimensional polytope can have $\Theta(n^{\lfloor d/2 \rfloor})$ facets (for fixed d)—asymptotically as many facets as the polytope itself. This fact was first proven by Amenta and Ziegler [1, 2] using a different construction. The construction given here is easier to visualize.

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