Delaunay Mesh Generation

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Preface

The study of algorithms for generating unstructured meshes of triangles and tetrahedra began with mechanical and aeronautical engineers who decompose physical domains into grids for the finite element and finite volume methods. Very soon, these engineers were joined by surveyors and specialists in geographical information systems, who use meshes called "triangulated irregular networks" to interpolate altitude fields on terrains, in the service of mapmaking, contouring, and visualization of topography. More recently, triangular meshes of surfaces have become prevalent as geometric models in computer graphics. These three groups of customers are the most numerous, but hardly the only ones; triangulations are found in most applications of multivariate interpolation.

Unfortunately, it is fiendishly hard to implement a reliable mesh generator. The demands on a mesh are heavy: it must conform to the geometric domain being modeled; it must contain triangles or tetrahedra of the correct shapes and sizes; it may have to grade from very short edges to very long ones over a short distance. These requirements are sometimes contradictory or even impossible. Most mesh generators are fragile, and sometimes fail when presented with a difficult domain, such as an object with many sharp angles or strangely curved boundaries.

One of the most exciting developments in computational geometry during the last several decades is the development of *provably good* mesh generation algorithms that offer guarantees on the quality of the meshes they produce. These algorithms make it easier to trust in the reliability of meshing software in unanticipated circumstances. Most mesh generators fall into one of three classes: advancing front mesh generators, which pave a domain with triangles or tetrahedra, laying down one at a time; meshers that decompose a domain by laying a grid, quadtree, or octree over it; and Delaunay mesh generators, which maintain a geometric structure called the *Delaunay triangulation* that has remarkable mathematical properties. To date, there are no provably good advancing front methods, and Delaunay meshers have proven to be more powerful and versatile than grid and octree algorithms, especially in their ability to cope with complicated domain boundaries.

In the past two decades, researchers have made progress in answering many intricate questions involving mesh generation: Can a mesher work for all input domains, including those with curved boundaries and sharp edges? If not, when and where must it make compromises? How accurately can a mesh composed of linear triangles or tetrahedra approximate the shape and topology of a curved domain? What guarantees can we make about the shapes and sizes of those triangles or tetrahedra? As a community, we now have algorithms that can tackle complex geometric domains ranging from polyhedra with internal boundaries to smooth surfaces to volumes bounded by piecewise smooth surfaces. And these algorithms come with guarantees.

This book is about algorithms for generating provably good Delaunay meshes, with an emphasis on algorithms that work well in practice. The guarantees they offer can include well-shaped triangles and tetrahedra, a reasonably small number of those triangles and tetrahedra, edges that are not unnecessarily short, topologically correct representations of curved domains, and geometrically accurate approximations of curved domains. As a foundation for these algorithms, the book also studies the combinatorial properties of Delaunay triangulations and their relatives, algorithms for constructing them, and their geometric and topological fidelity as approximations of smooth surfaces. After setting out the basic ideas of Delaunay mesh generation algorithms, we lavish attention on several particularly challenging problems: meshing domains with small angles; eliminating hard-to-remove "sliver" tetrahedra; and generating meshes that correctly match the topology and approximate the geometry of domains with smooth, curved surfaces or surface patches.

We have designed this book for two audiences: researchers, especially graduate students, and engineers who design and program mesh generation software. Algorithms that offer guarantees on mesh quality are difficult to design, so we emphasize rigorous mathematical foundations for proving that these guarantees hold and providing the core theoretical results upon which researchers can build even better algorithms in the future. However, one of the glories of provably good mesh generation is the demonstrated fact that many of its algorithms work wonderfully well in practice. We have included advice on how to implement them effectively. Although we promote a rigorous theoretical analysis of these methods, we have structured the book so readers can learn the algorithms without reading the proofs.

An important feature of this book is that it begins with a primer on Delaunay triangulations and constrained Delaunay triangulations in two and three dimensions, and some of the most practical algorithms for constructing and updating them. Delaunay triangulations are central to computational geometry and have found hundreds, probably thousands, of applications. Later chapters also cover Voronoi diagrams, weighted Voronoi diagrams, weighted Delaunay triangulations, restricted Voronoi diagrams, and restricted Delaunay triangulations. The last is a generalization of Delaunay triangulations that permits us to mesh surfaces in a rigorous, reliable way. We believe that this book is the first publication to combine so much information about these geometric structures in one place, and the first to give so much attention to modern algorithms.

The book can be divided into three parts of nearly equal length. The first part introduces meshes and the problem of mesh generation, defines Delaunay triangulations and describes their properties, and studies algorithms for their construction. The second part gives algorithms for generating high-quality meshes of polygonal and polyhedral domains. The third part uses restricted Delaunay triangulations to extend the algorithms to curved surfaces and domains whose boundaries are composed of curved ridges and patches.

The first chapter begins by describing the goals of mesh generation and telling a brief history of research in the field. Then it formally defines triangulations as simplicial complexes, and it defines the domains that those triangulations triangulate as other types of complexes. Chapters 2–5 cover Delaunay triangulations, constrained Delaunay triangulations, and algorithms for constructing and updating them in two and three dimensions. Chapter 2 introduces Delaunay triangulations of sets of points in the plane, their properties, and the geometric criteria that they optimize. It also introduces *piecewise linear complexes* (PLCs) as geometric structures for modeling polygonal domains; and triangulations of PLCs, particularly *constrained Delaunay triangulations* (CDTs), which generalize Delaunay triangulations to enforce the presence of specified edges. Chapter 3 presents algorithms for constructing Delaunay triangulations and CDTs, specifically, the incremental insertion and gift-wrapping algorithms. Chapter 4 extends Delaunay triangulations to higher dimensions and reviews geometric criteria that Delaunay triangulations of all dimensions optimize, some of which govern the accuracy of piecewise linear interpolation over triangles and tetrahedra. Chapter 5 reprises the incremental insertion and gift-wrapping algorithms for constructing Delaunay triangulations and CDTs in three dimensions.

Chapter 6 kicks off the middle third of the book with a discussion of *Delaunay refinement algorithms* for generating provably good triangular meshes of PLCs in the plane. Chapter 7 is an interlude in which we return to studying geometric complexes, including Voronoi diagrams, weighted Voronoi diagrams, and weighted Delaunay triangulations, which arm us with additional power to mesh polyhedral domains with small angles, eliminate some particularly troublesome tetrahedra of poor quality known as *slivers*, and handle curved surfaces.

Chapters 8–11 study algorithms for constructing tetrahedral meshes of polyhedral domains represented by three-dimensional PLCs. Chapter 8 presents a straightforward extension of the two-dimensional Delaunay refinement algorithm to three-dimensional domains with no acute angles. Chapter 9 describes an algorithm, new with this book, that meshes PLCs with small angles by constructing a *weighted* Delaunay triangulation. Chapters 10 and 11 describe a *sliver exudation* technique for removing slivers from a Delaunay mesh, thereby providing a mathematical guarantee on the quality of the tetrahedra. Although this guarantee is weak, the algorithm's success in practice exceeds what the theory promises. In both of these chapters, we have substantially improved the results in comparison with the previously published versions.

The final third of the book is devoted to meshing curved surfaces. A piecewise linear mesh cannot exactly conform to a curved surface, so we develop tools in approximation theory and topology to help guarantee the fidelity of a mesh to an underlying curved surface.

Chapter 12 covers topological spaces, homeomorphisms, isotopies, manifolds, and properties of point samples on manifolds. Chapter 13 introduces *restricted Voronoi diagrams*, whose Voronoi cells lie on a manifold, and their dual complexes, *restricted Delaunay triangulations*. We study conditions under which a restricted Delaunay triangulation is a topologically correct and geometrically close representation of a manifold. Chapter 14 describes mesh generation algorithms for curved surfaces and for the volumes they enclose; the meshes are restricted Delaunay triangulations. Chapter 15 makes the difficult jump from smooth surfaces to piecewise smooth surfaces, represented by a very general input domain called a *piecewise smooth complex* (PSC). PSCs bring with them all the difficulties that arise with polyhedral domains, such as enforcing boundary conformity and handling small domain angles, and all the difficulties that arise with smooth surfaces, such as guaranteeing topological correctness and small approximation errors. The algorithms described in the last two chapters and their analyses are considerably improved since their original publication.

At the end of each chapter, we provide historical and bibliographical notes and citations to acknowledge the pioneers who introduced the ideas in each chapter and to reference related ideas and publications. We include exercises, some of which we have assigned in graduate courses on mesh generation or computational geometry. We also use exercises as a way to include many interesting topics and improvements that we did not have enough room to discuss in detail, and theorems we did not have room to prove.

This book would have been impossible without help and advice. We thank our students who implemented versions of many of the algorithms presented in this book and generated pictures of the meshes they produced. Tathagata Ray meshed polyhedra with acute angles and remeshed polygonal surfaces. Joshua Levine meshed piecewise smooth complexes. Kuiyu Li helped to generate some of the figures. The Computational Geometry Algorithms Library (CGAL) project offered us a wonderful platform on which many of our implementations were carried out.

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