What is a Good Linear Finite Element?

Interpolation, Conditioning, Anisotropy, and Quality Measures

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Two Communities:
Mesh Generation and Error Analysis

Mesh generation people:

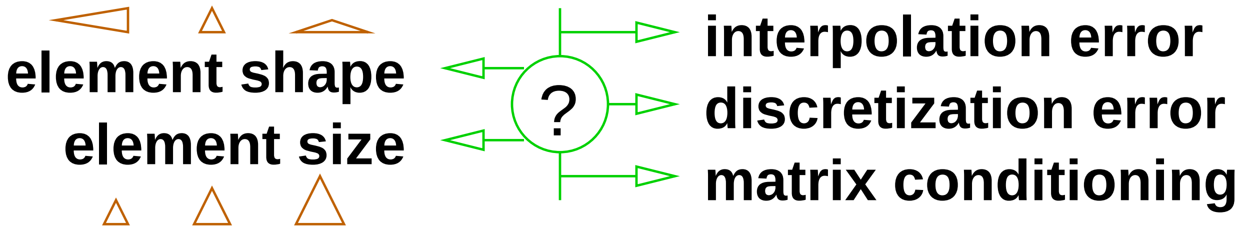
- Most don't *really* understand the goals of their own field!
- Know from experience small & large angles are bad (and are faintly aware why).

Numerical analysts:

- Tend to derive *asymptotic* error bounds & estimators (functional analysis, embedding theorems) – *not very useful* to meshing people!
- Meshing people can't read functional analysis anyway.

Error Bounds & Quality Measures

- The connections are still fuzzy.



(Especially in anisotropic cases.)

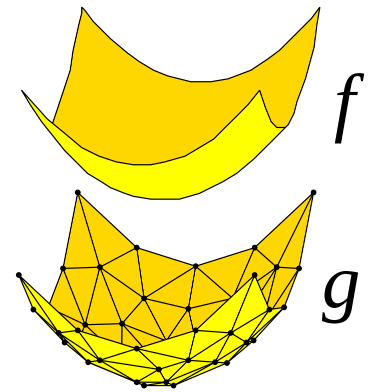
My goals:

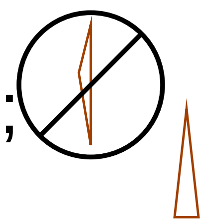
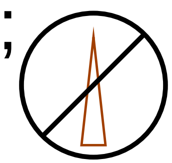
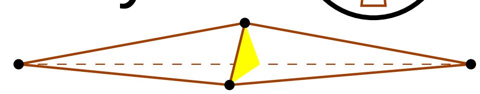
- (Nearly) tight bounds on worst−case errors, element stiffness matrix eigenvalues.
- Quality measures that can choose the better of two elements of intermediate quality.

  (Suitable for numerical optimization.)
- Guide mesh generators to make good elements.

Three Criteria for Linear Elements

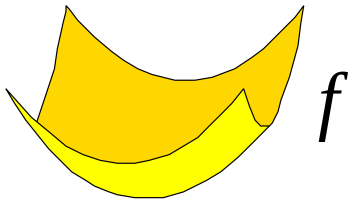
Let f be a function.
Let g be a piecewise linear interpolant of f over some triangulation.



Criterion	
Interpolation error $\|\| f - g \|\|_\infty$	Size very important. Shape only marginally important.
Gradient interpolation error $\|\| \nabla f - \nabla g \|\|_\infty$	Size important. Large angles bad; small okay. 
Element stiffness matrix maximum eigenvalue $\lambda_{\max}$	Small angles bad; large okay.  

Main Assumption

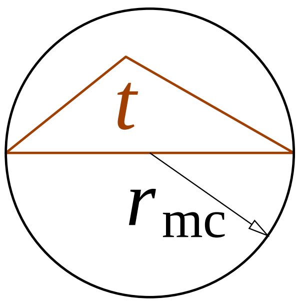
Curvature of f is bounded:



$$|f_{\mathbf{d}}''(p)| \le c$$

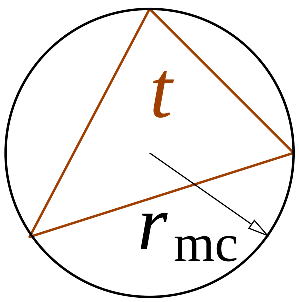
(second directional derivative along any direction $\mathbf{d}$)

Then over an element t,



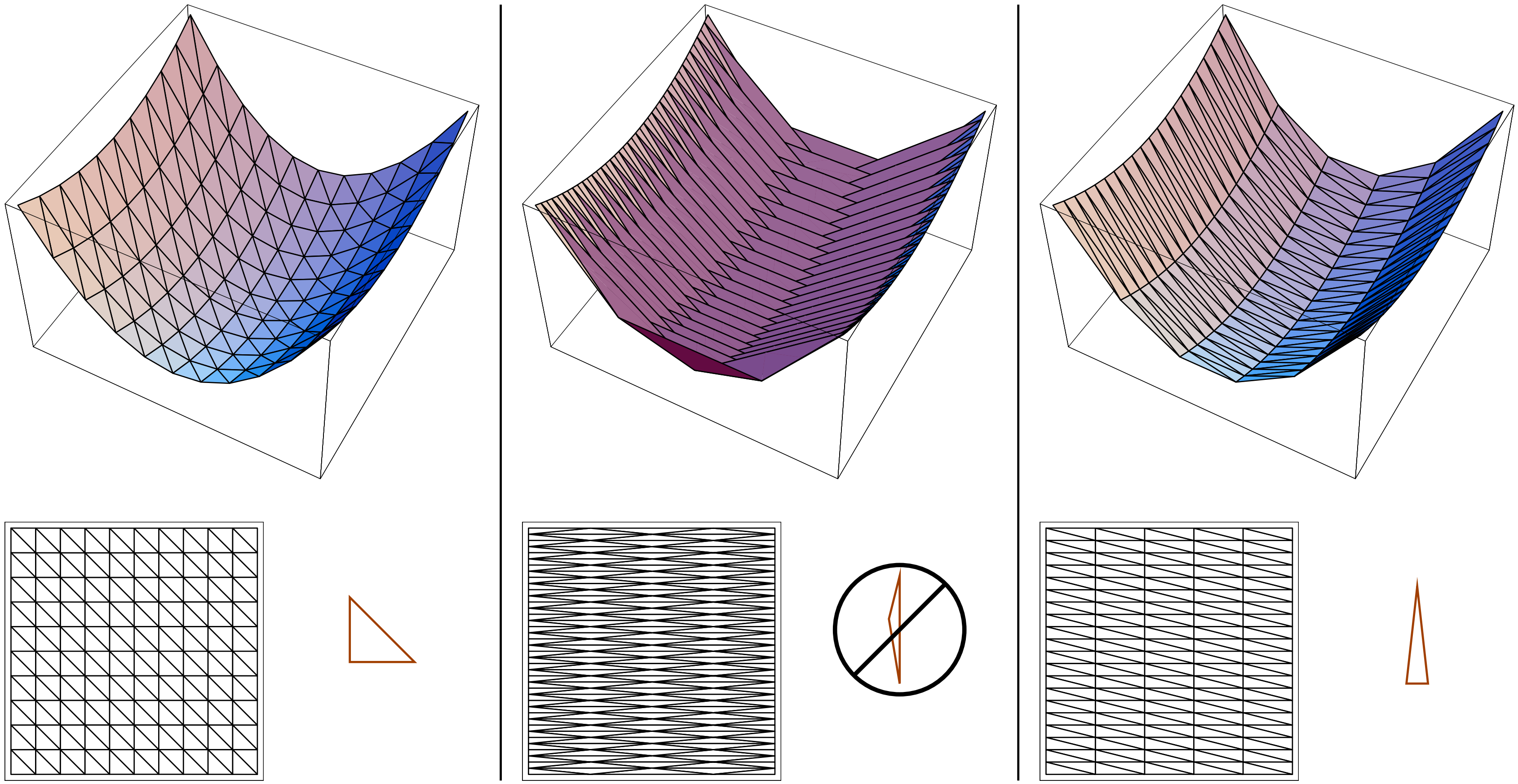
$$\| f - g \|_\infty \le \frac{c}{2} r_{\text{mc}}^2$$

where r_{mc} is the radius of the min−containment circle/sphere of t. [Waldron 1998.]



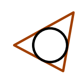
Sharp for triangles, tetrahedra, higher dimensions...

The Importance of Approximating Gradients Accurately

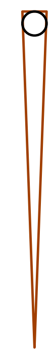


- $\| \nabla f - \nabla g \|_\infty$ affects discretization error in FEM.
- In mechanics, ∇f is the strains.

Classical Error Bounds on Gradients

- Approximation theory: error bound proportional to $l_{\text{max}}/r_{\text{in}}$. ← inradius of element  (See Johnson.)

  ↖ maximum edge length of element

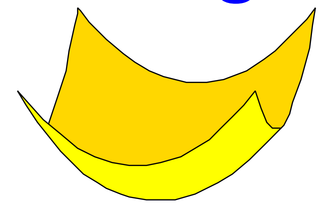


  - Not asymptotically tight − overestimates error for elements with small angles.

- Functional analysis: asymptotically tight error bound for triangles [Babuška and Aziz 1976]. And tetrahedra [Jamet 1976, Křížek 1992]?
  - But nobody knows the constant!

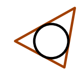
Error Bound: Gradients on Triangles

(Same assumption: bounded curvature.)



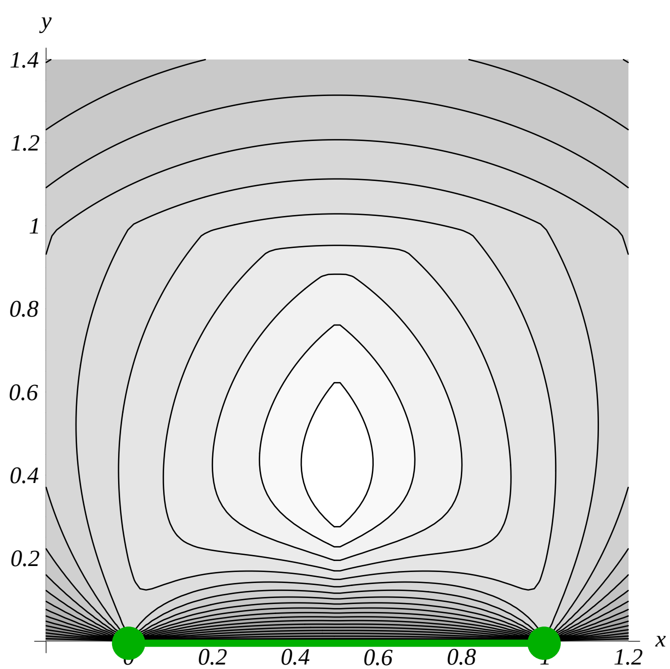
f

Over a triangle t,

edge lengths — inradius of t 

$$\|\nabla f - \nabla g\|_\infty \le c \, \frac{l_{\max}\, l_{\mathrm{med}} (l_{\min} + 4 r_{\mathrm{in}})}{4A}$$

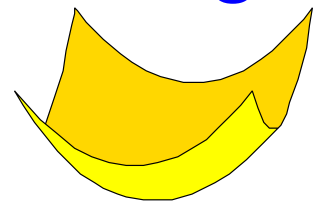
Area of t



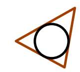
- Bound tight within factor of 2.

Error Bound: Gradients on Triangles

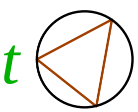
(Same assumption: bounded curvature.)

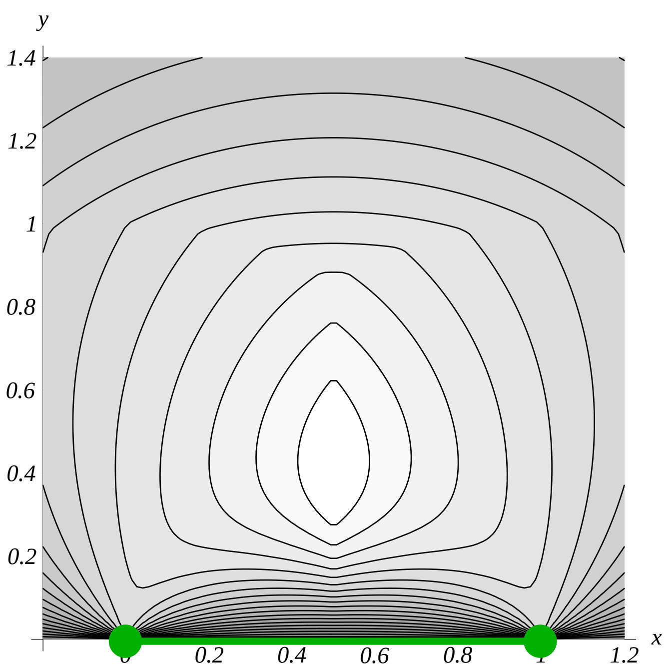


Over a triangle *t*,

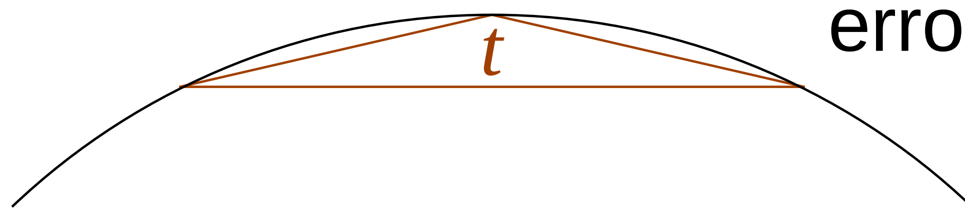
edge lengths — inradius of *t* 

$$\|\nabla f - \nabla g\|_\infty \le c\,\frac{l_{\max}\, l_{\mathrm{med}}(l_{\min} + 4\,r_{\mathrm{in}})}{4A} < 3\,c\,r_{\mathrm{circ}}$$

Area of *t* — Circumradius of *t* 



- Bound tight within factor of 2.
- Angle near 180° → large circumradius → large error.



Error Bound: Gradients on Tetrahedra

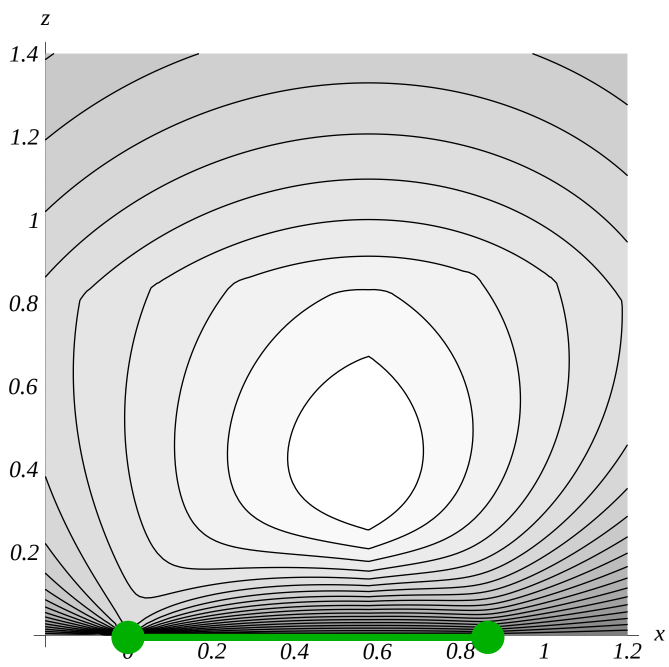
Over a tetrahedron t,

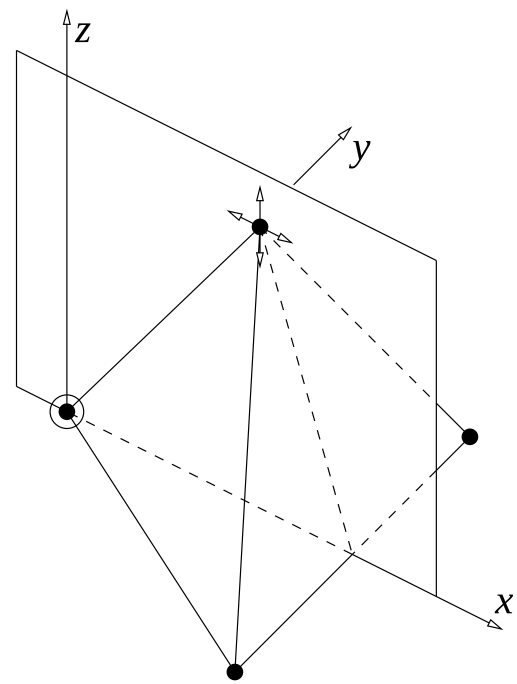
Edge lengths of t

$$\|\nabla f - \nabla g\|_\infty \le c \, \frac{\frac{1}{6V}\sum_{1 \le i \le j \le 4} A_i A_j l_{ij} + \max_i \sum_{j \ne i} A_j l_{ij}}{\sum_{m=1}^{4} A_m}$$

Volume of t

Face areas of t

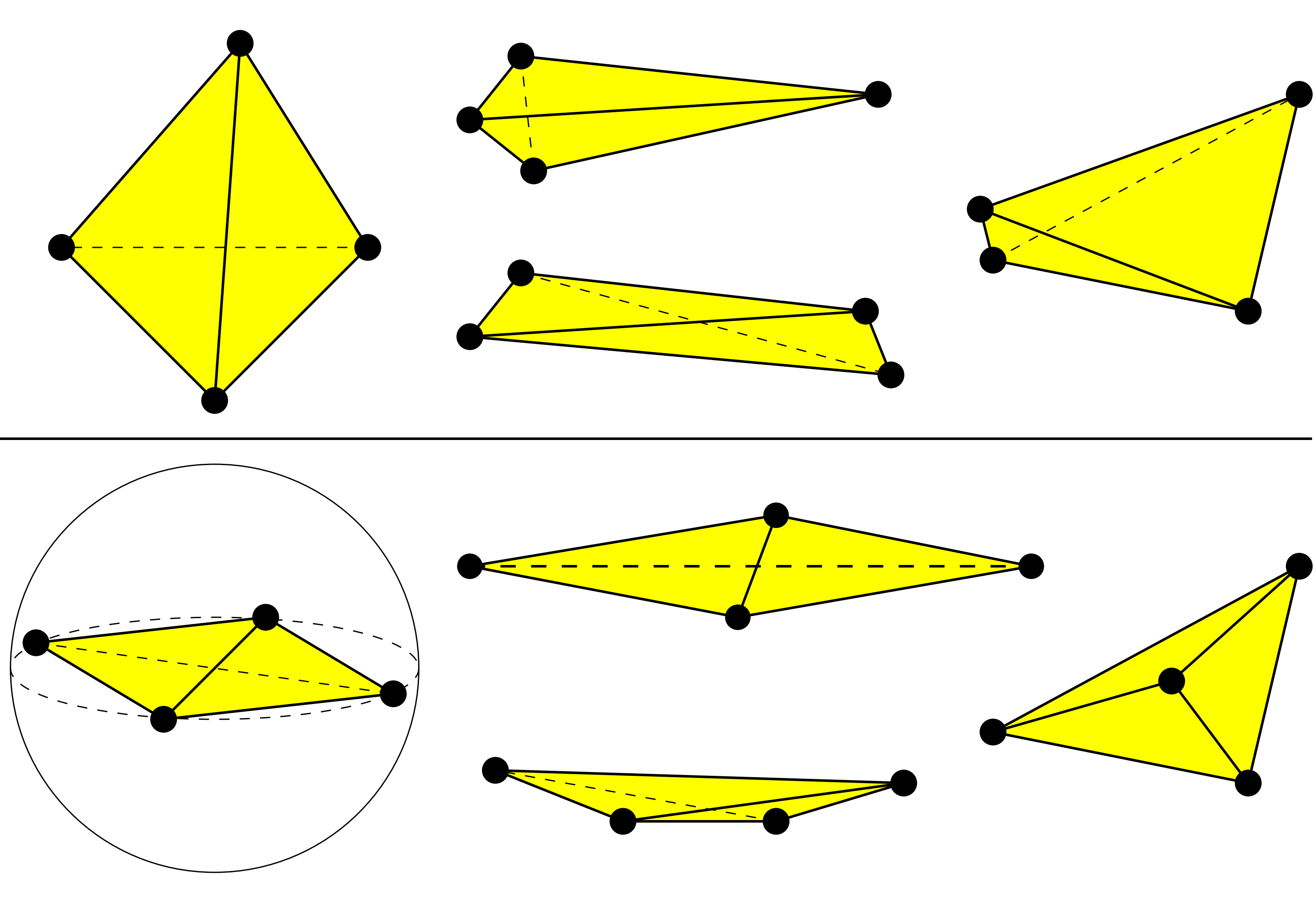




Good and Bad Tetrahedra for Interpolation

Good

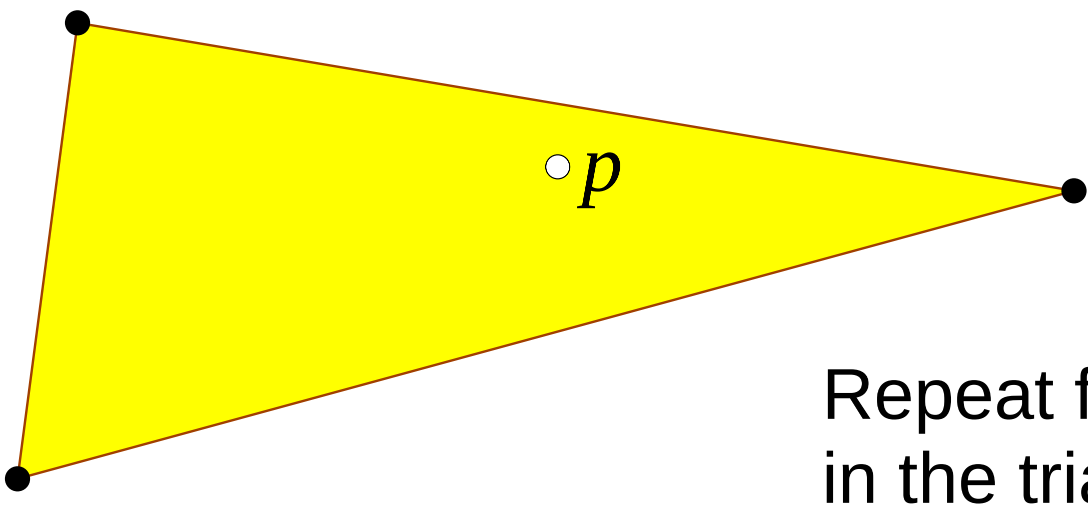
Bad



Deriving the New Gradient Error Bounds

Start with standard approximation theory:

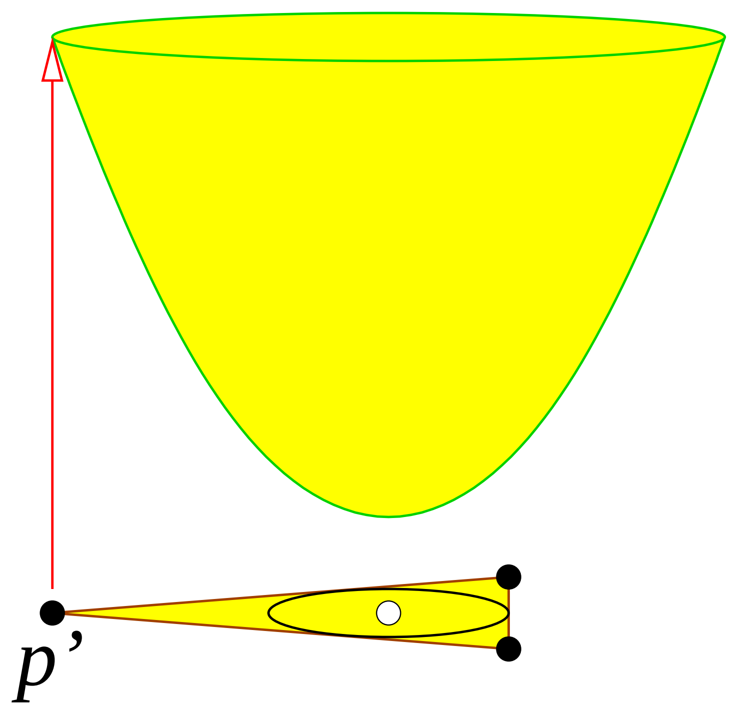
- Choose a point p.
- Take Taylor expansion of $f-g$ about p.
- Set it to zero at element vertices ($d+1$ equations).
- Eliminate $f(p)-g(p)$ term from equations.
- Curvature bounds yield naive bound on $\|\nabla f(p) - \nabla g(p)\|_\infty$.



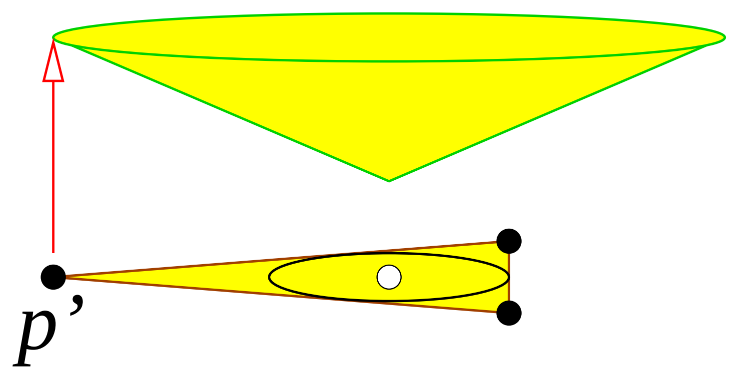
Repeat for each point p in the triangle...

Deriving the New Gradient Error Bounds

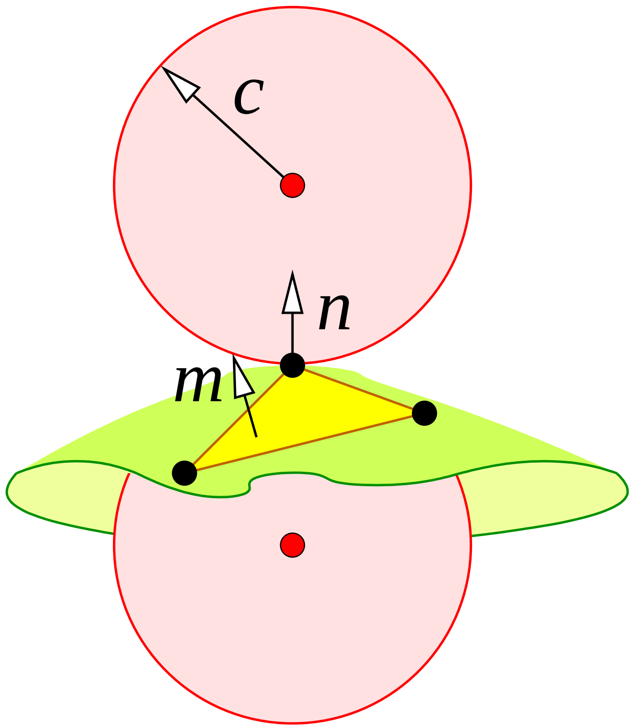
Naive bound on $\|\nabla f(p) - \nabla g(p)\|_\infty$ is parabolic. Worst point p' gives standard l_{max}/r_{in} bound.



But the naive error bound is minimized at the incenter. Curvature of f is bounded, so gradient of $\|\nabla f - \nabla g\|_\infty$ is bounded.



Error Bound: Triangle Normals on Surfaces



How much can triangle normal m deviate from surface normal n?

Assumption: spheres tangent to surface with radius c do not enclose any portion of surface.

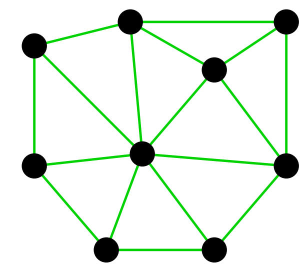
Angle between m and n (at any vertex) is at most

$$\alpha + \arcsin\left(\frac{2}{\sqrt{3}} \sin 2\alpha\right) + \arcsin \frac{l_{\text{med}}}{2c}, \qquad \alpha = \arcsin \frac{r_{\text{circ}}}{c}.$$

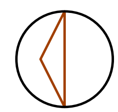
[Amenta, Choi, Dey, Leekha 2002.]

Delaunay Optimality

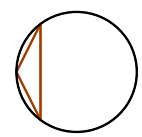
A set of vertices has many triangulations.



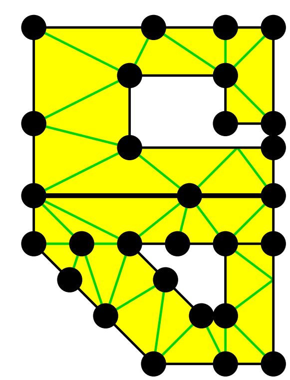
- In any dimensionality, the Delaunay triangulation minimizes the largest r_{mc}.



- In two dimensions, the Delaunay triangulation minimizes the largest r_{circ}.



A domain has many triangulations that respect its boundaries. Among these, the constrained Delaunay triangulation is optimal.



Conditioning of Global Stiffness Matrix

λ_{max}:

- Dominated by the single worst element.
- Depends on shape of element(s).
- 2D: Independent of element size.
- 3D: Largest element usually dominates.

λ_{min}:

- Relatively independent of shape.
- Directly proportional to areas/volumes of elements.
- Somewhere between smallest and largest elements (times a constant).

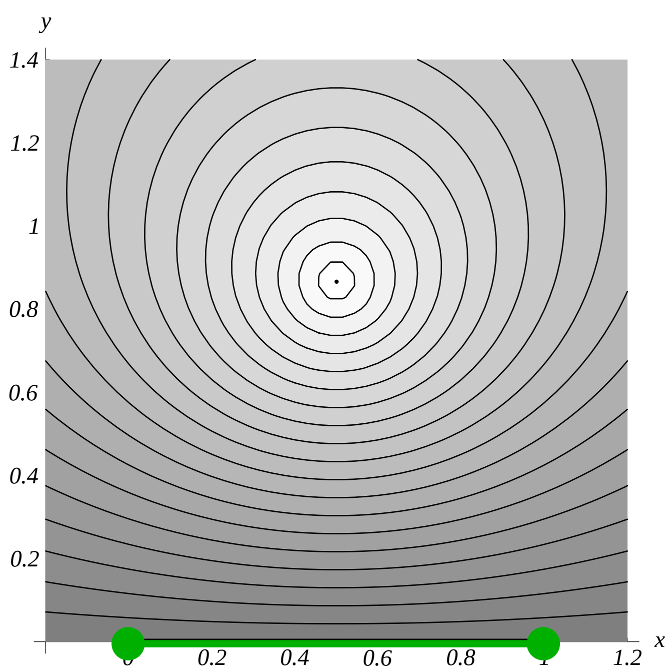
Conditioning: Maximum Eigenvalue of Element Stiffness Matrix

(for Poisson's Equation)

edge lengths

$$\lambda_{\max} = \frac{l_1^2 + l_2^2 + l_3^2 + 2\sqrt{(l_1^2 + l_2^2 + l_3^2)^2 - 48A^2}}{8A}$$

Area of t

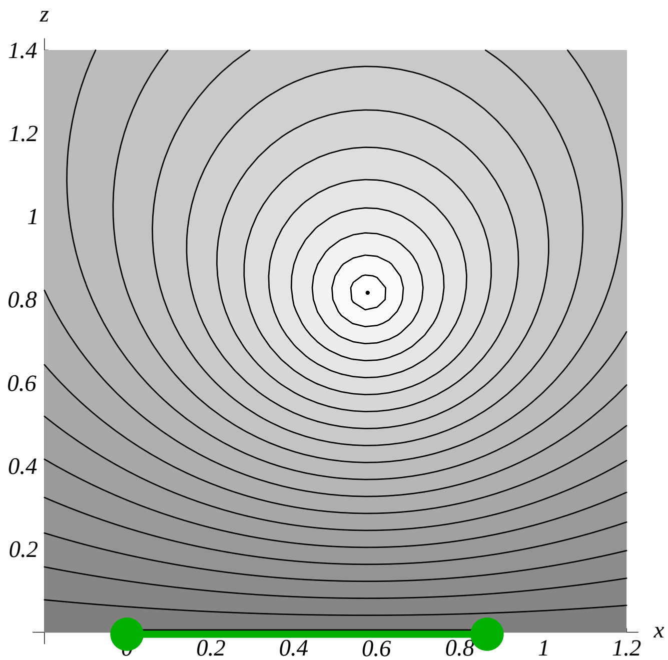


- Maximum eigenvalue is a quality measure that prefers equilateral triangles.
- Small angles are deleterious.

Maximum Eigenvalue in 3D

(for Poisson's Equation)

- Eigenvalue for tetrahedron requires solving a cubic equation.
- Eigenvalue smallest for equilateral tetrahedra.

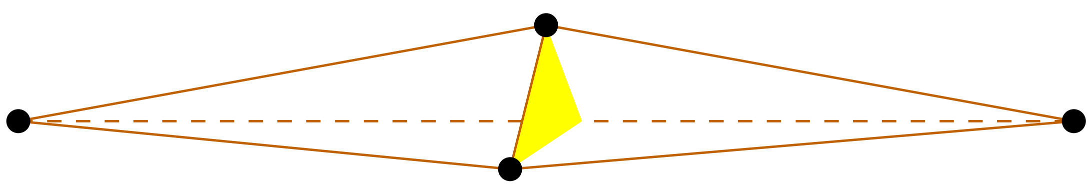


- Dihedral angles, not planar angles, are related to quality.
- It's a "well−known fact" that both small and large dihedral angles hurt conditioning...

WRONG!!!

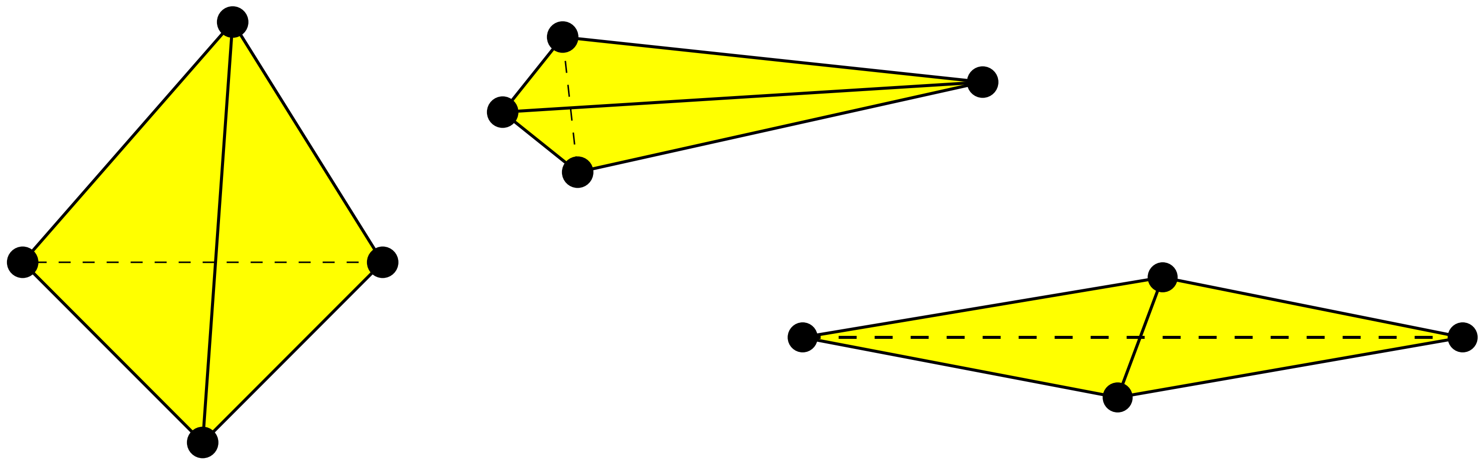
Surprise #1

A tetrahedron can have a dihedral angle arbitrarily close to 180° with no dihedral smaller than 60°. Such a tetrahedron does not hurt conditioning at all!

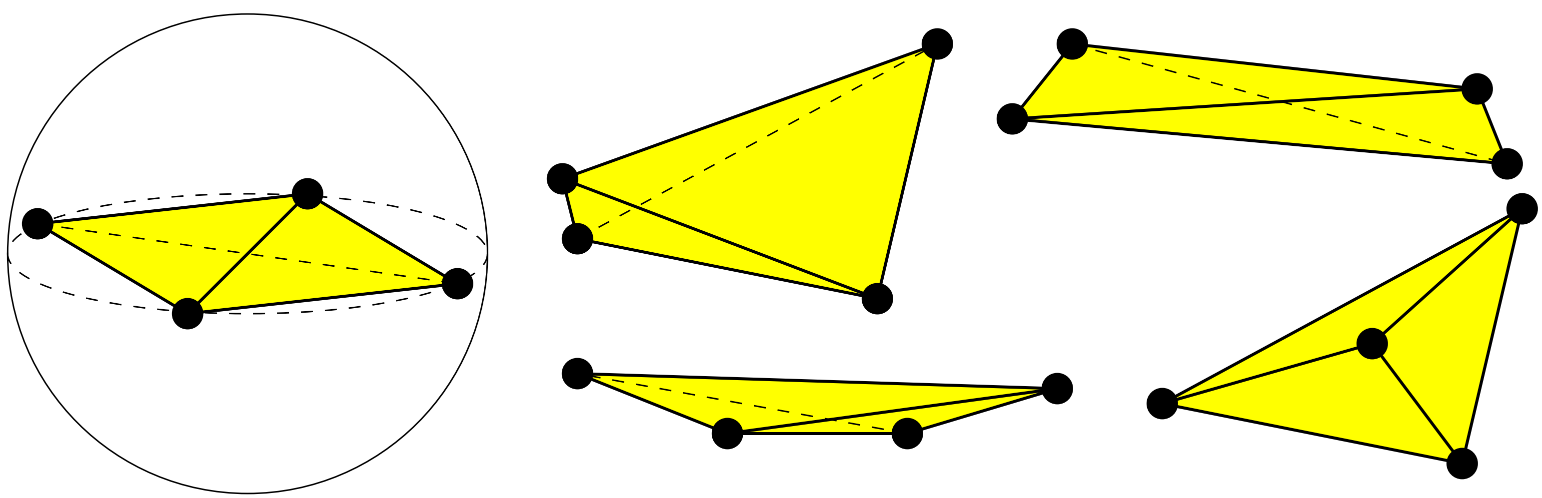


Good and Bad Tetrahedra for Conditioning

Good

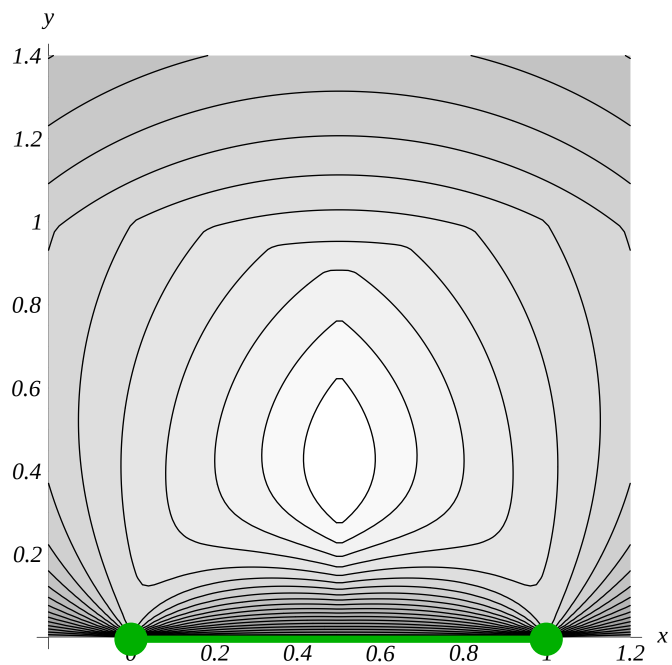


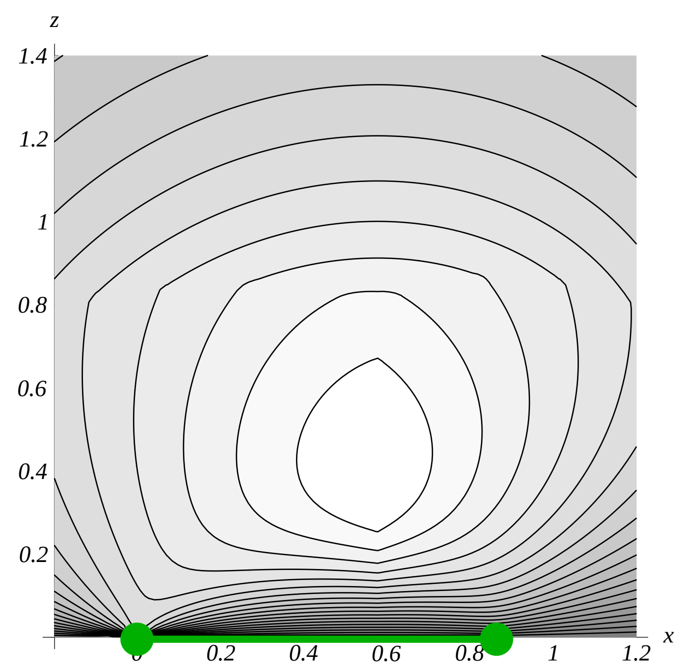
Bad



Quality Measures

- Used to evaluate & choose elements.
- Reciprocal of interpolation error or max eigenvalue.
- Behave well as objective functions for mesh smoothing by numerical optimization.



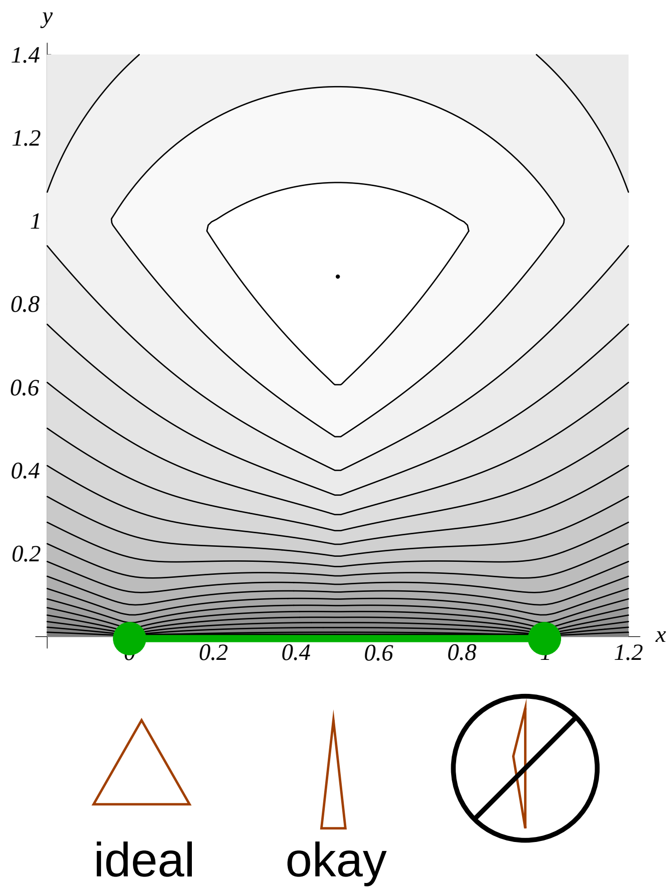


triangles $\|\nabla f - \nabla g\|_\infty$ tetrahedra

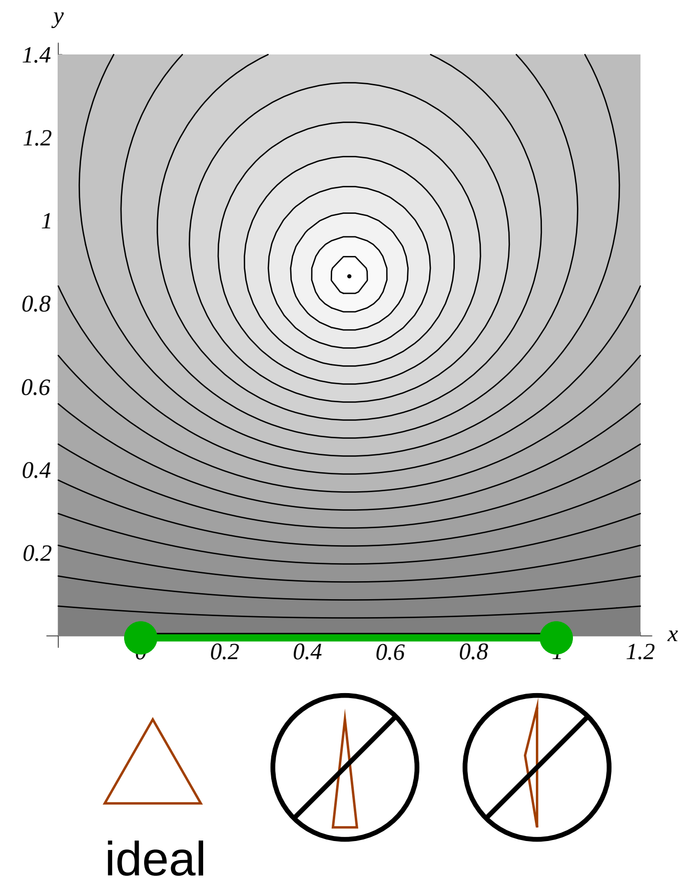
Scale−Invariant Quality Measures

...measure how effective an element's shape is for a fixed amount of area.

$$\|\nabla f - \nabla g\|_\infty$$



$$\lambda_{\max}$$



Two Types of Quality Measures

Scale−invariant quality measures:

- Separate the effects of shape from the effects of size.
- Popular in mesh generation because they’re easy to understand.

Size−and−shape measures & error bounds:

- Incorporate the effects of shape and size in one number.
- Require more understanding of application.
- What I advocate for most purposes.

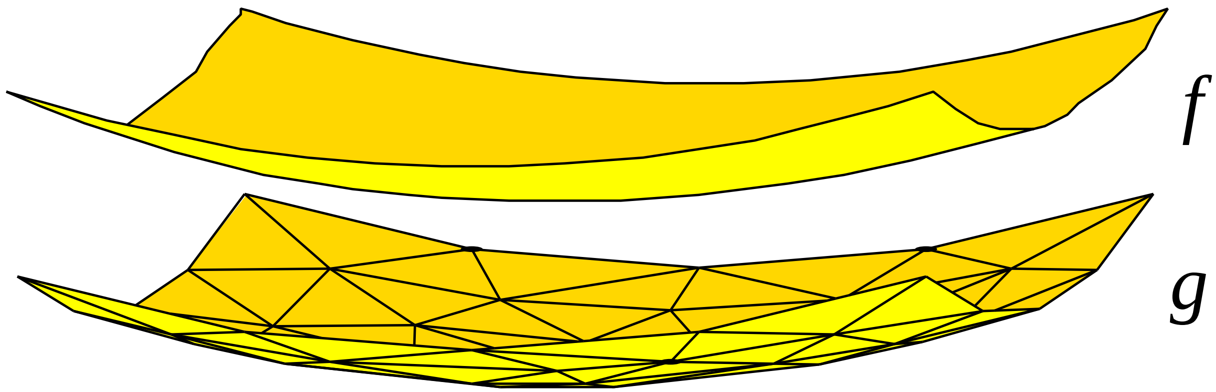
Why Scale−Invariant Measures are Misleading

- Interpolation/discretization error and (in 3D) conditioning depend on element size too!
- It's okay (usually) for smaller elements to be worse shaped than big ones. Smoothing, cleanup, Delaunay refinement should take this into account.
- Example: 2D scale−invariant measure for interpolated gradients suggests ‘‘minimize large angles.’’ But size−and−shape measure says ‘‘minimize circumradii’’ → Delaunay.

Uses of Error Bounds & Quality Measures

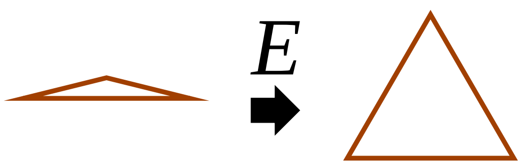
- **Mesh refinement:** Refine element if either $\| f - g \|_\infty$ or $\| \nabla f - \nabla g \|_\infty$ is too large. Use Delaunay refinement if conditioning/shape bad.
- **Mesh smoothing:** Quality measures are designed for numerical optimization of vertex positions. Smoother, slightly weaker measures available.
- **Topological mesh improvement:** Use quality measures plus refinement bound to judge elements.
- **Vertex placement in advancing front meshing.**

Anisotropy and Interpolation Error

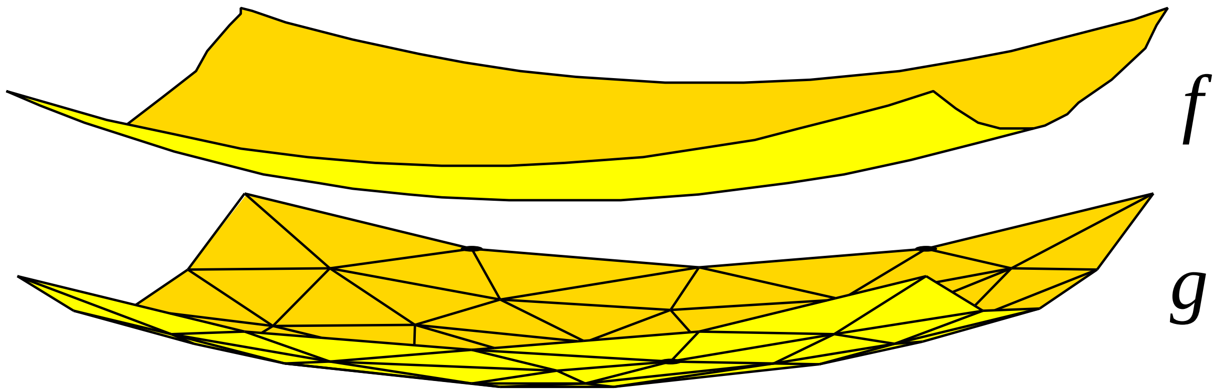


H = Hessian of f. Let $E^2 = H$ with E symmetric pos−def.

You can judge the error $\| f - g \|_\infty$ of an element t by judging Et by isotropic error bounds/measures.

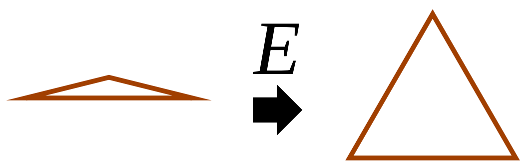


Anisotropy and Gradient Interpolation



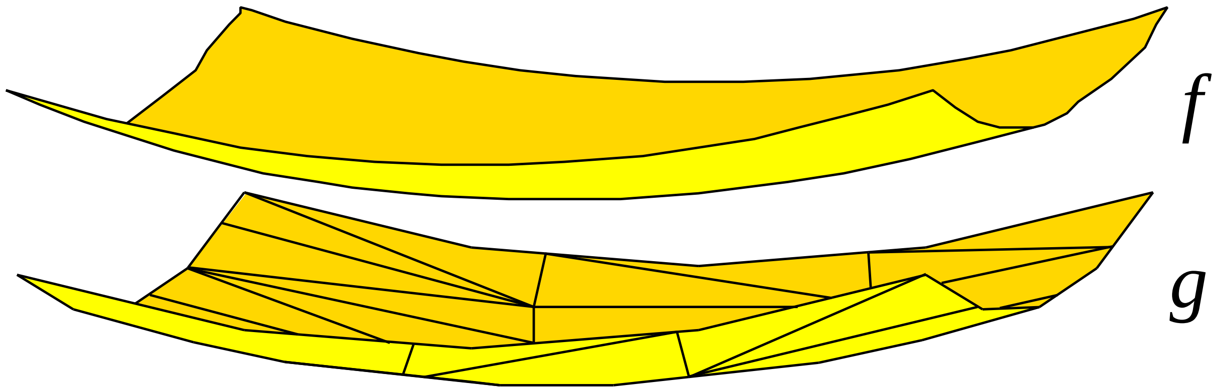
Large angles are fine if aligned correctly and not *too* large. 

A good element t for controlling $\|\nabla f - \nabla g\|_\infty$ is one for which Et has no large angle.



But...

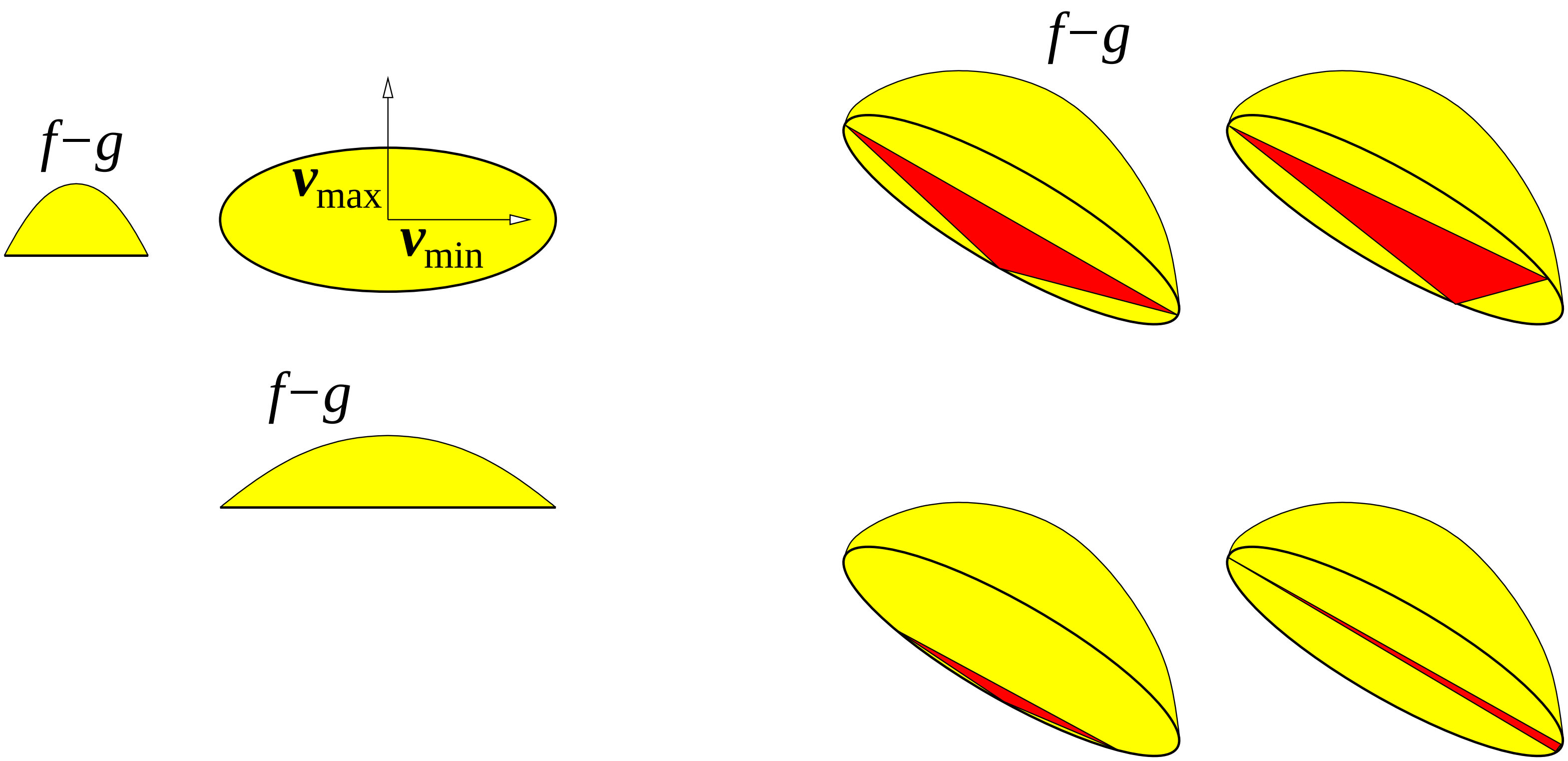
Surprise #2: Superaccurate Gradients



Longer, thinner elements than expected sometimes give the best accuracy! (For a fixed # of elements.)

But no large angles allowed for these extra−long ones. And they must be very precisely aligned.

Superaccurate Gradients

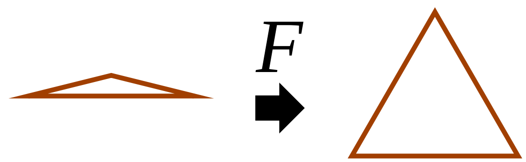


Anisotropy and Conditioning

Ideal element for stiffness matrix depends on anisotropy of the PDE, not the solution.

$$-\nabla \cdot B \nabla f = 0$$

$$F^2 = B^{-1}$$



Large and small angles are fine if Ft looks good by isotropic standards. 

Equilateral elements can be quite bad for conditioning.

Surprise #3: Anisotropy Blues

- Interpolation and conditioning don’t always agree on the ideal aspect ratio or orientation!
- Superaccuracy implies that discretization error sometimes disagrees with both interpolation and conditioning!

Advice:

- Interpolation/discretization error can always be improved by refining. Conditioning cannot.
- With finite elements, always choose small discretization error over interpolation.

Concluding Questions

- What about quadratic triangles and tetrahedra?
- What about bilinear quads and trilinear hexes?
- What if the curvature bound is in the L_2 norm?