Streaming Construction of Delaunay Triangulations

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Joint work with Martin Isenburg, Yuanxin Liu, & Jack Snoeyink
Streaming Construction of Delaunay Triangulations

Or, “How we compute supercomputer-sized Delaunay triangulations on an ordinary laptop computer.”
Every point set has a Delaunay triangulation. Think of it as a function that takes a set of points and outputs a triangulation.
The Delaunay Triangulation

...is a triangulation whose triangles are all *Delaunay*.

An triangle is Delaunay if it has an empty circumscribing circle – one that encloses no vertex.
The Delaunay Triangulation

The circumcircle of every Delaunay triangle is empty.
Applications of Triangulations

Contouring

Rendering

Terrain Databases and Geographical Information Systems
Elevation Collection by LIDAR
Related Work

Related Work


- Agarwal, Arge, and Yi [2005] implemented an external memory 2D triangulator based on randomized divide-and-conquer. (1 billion triangles on desktop machine.)
Related Work


- Agarwal, Arge, and Yi [2005] implemented an external memory 2D triangulator based on randomized divide-and-conquer. (1 billion triangles on desktop machine.)

- Blandford, Blelloch, and Kadow [2006] implemented a 3D parallel triangulator. (10 billion tetrahedra on 64 processors.)
Streaming Delaunay
Streaming Computation

A restricted version of out–of–core computation.

- Algorithm makes one (or a few) pass(es) over input stream, and writes output stream.
Streaming Computation

A restricted version of out–of–core computation.

- Algorithm makes one (or a few) pass(es) over input stream, and writes output stream.
- Processes data in memory buffer much smaller than the stream sizes.
Streaming Computation

A restricted version of out-of-core computation.

- Algorithm makes one (or a few) pass(es) over input stream, and writes output stream.
- Processes data in memory buffer much smaller than the stream sizes.
- Nothing is stored temporarily to disk.
Advantages of Streaming

- Streaming tools can run concurrently in a pipeline.

- Often much faster than other out-of-core algorithms!
Our Accomplishments

- 9 billion triangles in 6.6 hours on a laptop.
Our Accomplishments

- 9 billion triangles in 6.6 hours on a laptop.
- 12 times faster than Agarwal–Arge–Yi.
Our Accomplishments

• 9 billion triangles in 6.6 hours on a laptop.

• 12 times faster than Agarwal–Arge–Yi.

• 800 million tetrahedra in under 3 hours.
Algorithm Choice

The *incremental insertion* algorithm for constructing Delaunay triangulations.
Algorithm Choice

The *incremental insertion* algorithm for constructing Delaunay triangulations.

Why?

- We have little control over the order of points in the input stream.
Algorithm Choice

The \textit{incremental insertion} algorithm for constructing Delaunay triangulations.

Why?

- We have little control over the order of points in the input stream.
- We modified existing, sequential Delaunay triangulation codes (2D & 3D) for streaming.
Bowyer–Watson Algorithm

Insert one vertex at a time.
Bowyer–Watson Algorithm

Insert one vertex at a time.
Remove all triangles/tetrahedra that are no longer Delaunay.
Bowyer–Watson Algorithm

Insert one vertex at a time.
Remove all triangles/tetrahedra that are no longer Delaunay.
Retriangulate the cavity with a fan around the new vertex.
Spatial Finalization
Spatial Finalization

- Subdivide space into *finalization regions.*
Spatial Finalization

- Subdivide space into *finalization regions*.

- Inject into the stream *spatial finalization tags* that indicate “there are no more points in this region.”
Spatial Finalization: Why?

Triangles whose circumscribing circles lie entirely in finalized space can be written to disk immediately.
Triangles whose circumscribing circles intersect an unfinalized region remain in memory.
Streaming Delaunay Pipeline

Finalizer

Triangulator
The Finalizer
How to Finalize

1. Compute bounding box.
How to Finalize

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1. Compute bounding box.
How to Finalize

1. Compute bounding box.
2. Finalization grid.
How to Finalize

1. Compute bounding box.
2. Finalization grid.
   - Count points.
# How to Finalize

1. Compute bounding box.
2. Finalization grid.
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1. Compute bounding box.
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   - Count points.
   - Store “sprinkle points.”
How to Finalize

1. Compute bounding box.
2. Finalization grid.
   - Count points.
   - Store “sprinkle points.”
3. Output finalized points.
   - Release chunks.
   - Inject finalization tags.
### How to Finalize

1. Compute bounding box.
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3. Output finalized points.
   - Release chunks.
   - Inject finalization tags.

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Streaming Point Format

```
# npoints 22514
# bb_min_f 0 0 192.25
# bb_max_f 122.5 158.5 215.7
# datatype SP_FLOAT
v 0 3 193.1
v 0.4375 3.5 193.11
v 0.59375 4 193.09
v 0.6875 4.5 193.11
v 0.90625 2.5 193.13
v 0.90625 4.5 193.07
x cell 2732
v 0 7.5 192.91
v 0.21875 9 192.93
v 0.53125 9 192.99
v 0.8125 7.5 192.96
v 0.9375 9 193.04
v 1.03125 9.5 193.06
v 1.65625 8.5 193.04
x cell 2740
```

chunk of points

chunk of points

finalization tag

finalization tag
Spatial Coherence

The correlation between the proximity of points in space and their proximity in the stream.
Spatial Coherence

The correlation between the proximity of points in space and their proximity in the stream.
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The correlation between the proximity of points in space and their proximity in the stream.
Danger: Potential Quadratic Time
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Point Reordering by Finalizer

- **Chunking**: Buffering points in a region until all points arrive, then releasing them together.
  - Points in a chunk are randomly reordered.
Point Reordering by Finalizer

- **Chunking**: Buffering points in a region until all points arrive, then releasing them together.
  - Points in a chunk are randomly reordered.

- **Sprinkling**: Promoting a small random sample of “sprinkle points” to earlier in the stream.
  - Circumvents danger of quadratic behavior.
Streaming Delaunay Triangulation

When point arrives:
- Locate enclosing triangle.
- Update triangulation.

Finalized space

Regions not yet finalized
Streaming Delaunay Triangulation

When point arrives:
- Locate enclosing triangle.
- Update triangulation.

When tag arrives:
- Identify final triangles.
- Write them out & free their memory.

Regions not yet finalized
Streaming Delaunay Triangulation

When point arrives:
- Locate enclosing triangle.
  - Start at newest triangle.
Streaming Delaunay Triangulation

When point arrives:
• Locate enclosing triangle.
  ○ Start at newest triangle.
  ○ Walk to point.
Streaming Delaunay Triangulation

When point arrives:
- Locate enclosing triangle.
  - Start at newest triangle.
  - Walk to point.
Streaming Delaunay Triangulation

When point arrives:
- Locate enclosing triangle.
  - Start at newest triangle.
  - Walk to point.
  - If that fails, restart from cell triangle.
  - If that fails, do exhaustive search.
Streaming Delaunay Triangulation

When point arrives:
- Locate enclosing triangle.
- Update triangulation.
Streaming Delaunay Triangulation

When tag arrives:
- Identify final triangles.

Only these triangles need to be checked:
- New triangles (since last tag).
- Triangles waiting for this tag.
Streaming Delaunay Triangulation

When tag arrives:
• Identify final triangles.

Checked triangles that aren’t final are assigned a tag to wait for.
## Results

<table>
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<tr>
<th>name</th>
<th># of points file size</th>
<th>options</th>
<th>spdelaunay2d max active triangles</th>
<th>h:mm:ss disk</th>
<th>pipe</th>
<th>MB</th>
<th>output mesh # of triangles file size</th>
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<tr>
<td>neuse</td>
<td>500,141,313</td>
<td>l8e4</td>
<td>148,198</td>
<td>39:24</td>
<td>40:53</td>
<td>20</td>
<td>(single)</td>
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<tr>
<td>(double)</td>
<td>11.2 GB</td>
<td>l9e4</td>
<td>75,705</td>
<td>37:14</td>
<td>38:37</td>
<td>10</td>
<td>1,000,282,528</td>
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<td>l10e4</td>
<td>60,026</td>
<td>35:15</td>
<td>35:18</td>
<td>11</td>
<td>16.9 GB</td>
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# Results

<table>
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<tr>
<th>finalized input points</th>
<th>spdelaunay2d</th>
<th>output mesh</th>
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</thead>
<tbody>
<tr>
<td>name</td>
<td>max active triangles</td>
<td>h:mm:ss disk</td>
</tr>
<tr>
<td>file size</td>
<td></td>
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<tr>
<td>neuse (double)</td>
<td>500,141,313</td>
<td>l_8e_4</td>
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<td>60,026</td>
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</tbody>
</table>

<table>
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<th>depth (k)</th>
<th>time/pass, m:ss</th>
<th>MB</th>
<th>points buffered</th>
<th>occu’d cells</th>
<th>points per cell avg</th>
<th>max</th>
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<td>neuse</td>
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<td>5:56 5:56 8:32</td>
<td>94</td>
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<td>5:56 5:56 7:55</td>
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<tr>
<td>11.2 GB</td>
<td>10</td>
<td>5:56 5:56 6:49</td>
<td>61</td>
<td>1,518,675</td>
<td>306,334</td>
<td>1,632</td>
<td>6,544</td>
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</tbody>
</table>

- **Us:** 12 min finalize + 36 min triang. = 48 min.
- **Agarwal–Arge–Yi:**
  3 hr sort + 7.5 hr triang. = 10.5 hr.
## Results

<table>
<thead>
<tr>
<th>finalized input points</th>
<th>spdelaunay2d</th>
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</tr>
</thead>
<tbody>
<tr>
<td>name</td>
<td># of points</td>
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<tr>
<td>neuse</td>
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<td>3 x 3</td>
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<tr>
<td>(double)</td>
<td>4,501,271,817</td>
<td>78,475</td>
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107 min finalize + 278 min triang. = 6 hr 25 min. 425 MB finalize + 11 MB triang. = 436 MB.
3D Streaming Delaunay

815 million tetrahedra in 2:41 hrs & 795 MB. (Pre-finalized points.)
3D Streaming Delaunay

Does not work for surface scans. Circumspheres are too big.
3D Streaming Delaunay

Future work: spherical finalization regions from Delaunay triangulation of a random sample.
3D Streaming Delaunay

Future work: spherical finalization regions from Delaunay triangulation of a random sample.
Conclusions
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- Most real-world points sets have lots of spatial coherence → sorting doesn’t help.
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• Most real-world points sets have lots of spatial coherence → sorting doesn’t help.
• Like aikido: Use your opponent’s spatial coherence against him.
Conclusions

- Most real-world points sets have lots of spatial coherence, sorting doesn’t help.
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- Not an external memory algorithm! No temporary storage to disk.
Conclusions

- Most real-world points sets have lots of spatial coherence, sorting doesn’t help.
- Like aikido: Use your opponent’s spatial coherence against him.
- Not an external memory algorithm! No temporary storage to disk.
- 12 times faster than best previous 2D approach.
Conclusions

• Most real-world points sets have lots of spatial coherence → sorting doesn’t help.

• Like aikido: Use your opponent’s spatial coherence against him.

• Not an external memory algorithm!
  No temporary storage to disk.

• 12 times faster than best previous 2D approach.

• Streaming triangulation can be piped directly to another streaming application.
Streaming Digital Elevation Maps
Streaming Contour Extraction
Thanks

- Kevin Yi at Duke supplied the Neuse River Basin data.
- Martin Isenburg & Yuanxin “Leo” Liu did most of the programming.