A Condition Guaranteeing the Existence of Higher-Dimensional Constrained Delaunay Triangulations

Jonathan Richard Shewchuk
School of Computer Science
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213
jrs@cs.cmu.edu

Abstract

Let $X$ be a complex of vertices and piecewise linear constraining facets embedded in $E^d$. Say that a simplex is strongly Delaunay if its vertices are in $X$ and there exists a sphere that passes through its vertices but passes through and encloses no other vertex. Then $X$ has a $d$-dimensional constrained Delaunay triangulation if each $k$-dimensional constraining facet in $X$ with $k \leq d - 2$ is a union of strongly Delaunay $k$-simplices.

This theorem is especially useful in $E^3$ for forming tetrahedralizations that respect specified planar facets. If the bounding segments of these facets are subdivided so that the subsegments are strongly Delaunay, then a constrained tetrahedralization exists. Hence, fewer vertices are needed than in the most common practice in the literature, wherein additional vertices are inserted in the relative interiors of facets to form a conforming (but unconstrained) Delaunay tetrahedralization.

1 Introduction

Many applications can benefit from triangulations that have properties similar to Delaunay triangulations, but are constrained to contain specified edges or faces. For instance, Delaunay triangulations have desirable properties when used for function interpolation, but a triangulation might be required to conform to specified facets so that discontinuities can be represented. Delaunay triangulations can also serve as meshes that represent objects for rendering or the numerical solution of partial differential equations. In these cases, each triangulation is required to conform to the shape of the object being modeled.

In two dimensions, there are two popular alternatives for creating a Delaunay-like triangulation that conforms to constraints. In either case, the input is a planar straight line graph (PSLG) $X$, which is a set of vertices and segments (constraining edges) as illustrated in Figure 1 (upper left). A PSLG is required to contain both endpoints of every segment it contains, and a segment may intersect vertices and other segments only at its endpoints. A triangulation is sought that includes the vertices of $X$ and respects the segments of $X$.

The first alternative is to form a conforming Delaunay triangulation (Figure 1, lower left). The vertices of $X$ are augmented by additional vertices (sometimes called Steiner points) carefully chosen so that the Delaunay triangulation of the augmented vertex set conforms to all the segments—in other words, so that each segment is represented by a contiguous linear sequence of edges of the triangulation. Edelsbrunner and Tan [4] show that any PSLG can be triangulated with the addition of no more than $O(m^2 n)$ augmenting vertices, where $m$ is the number of segments in $X$, and $n$ is the number of vertices. Not every PSLG requires this many augmenting vertices, but the numbers required in practice may nonetheless seem undesirably large for some applications.

The second alternative is to form a constrained Delaunay triangulation (CDT) [1] (Figure 1, lower right). A CDT of $X$ has no vertices not in $X$, and every segment of $X$ is a single edge of

Figure 1: The Delaunay triangulation (upper right) of the vertices of a PSLG (upper left) might not respect the segments of the PSLG. These segments can be incorporated by adding vertices to obtain a conforming Delaunay triangulation (lower left), or by forgoing Delaunay triangles in favor of constrained Delaunay triangles (lower right).
the CDT. However, a CDT is not a Delaunay triangulation. In an
ordinary Delaunay triangulation, every simplex (of any dimension-
ality) is Delaunay. A simplex is Delaunay if its vertices are in \( X \)
and there exists a circumsphere of the simplex—a circle that passes
through all its vertices—that encloses no vertex of \( X \) (although any
number of vertices is permitted on the circle itself). In a CDT, this
requirement is waived, and instead every simplex must either be a
segment specified in \( X \) or be constrained Delaunay. A simplex is
constrained Delaunay if it has a circumsphere that encloses no ver-
tex of \( X \) that is visible from any point in the relative interior of
the simplex; and furthermore, the relative interior of the simplex does
not intersect any segment. Visibility is occluded only by segments of
\( X \).

Figure 2 demonstrates examples of a constrained Delaunay edge
e and a constrained Delaunay triangle \( t \). Input segments appear as
bold lines. Although there is no empty circle that contains \( e \), the de-
picted circumsphere of \( e \) encloses no vertex that is visible from the
relative interior of \( e \). There are two vertices inside the circle, but
both are hidden behind segments. Hence, \( e \) is constrained Dela-
uay. Similarly, the circumsphere of \( t \) contains two vertices, but both
are hidden from the interior of \( t \) by segments, so \( t \) is constrained
Delaunay.

The advantage of a CDT over a conforming Delaunay triangula-
tion is that it has no vertices other than those in \( X \). However,
a conforming Delaunay triangulation’s triangles are Delaunay, but
those of a CDT are not. Nevertheless, a CDT retains many of
the desirable properties of Delaunay triangulations. For instance,
a two-dimensional CDT maximizes the minimum angle in the tri-
angulation, compared with all other constrained triangulations of
\( X \) [7].

Unfortunately, CDTs have not been generalized to dimensions
higher than two. One reason is that in three or more dimensions,
there are polytopes that cannot be triangulated at all without addi-
tional vertices. Schönhardt [12] furnishes a three-dimensional ex-
ample depicted in Figure 3 (right). The easiest way to envision this
polyhedron is to begin with a triangular prism. Imagine grasping
the prism so that one of its two triangular faces cannot move, while
the opposite triangular face is rotated slightly about its center with-
out moving out of its plane. As a result, each of the three square
faces is broken along a diagonal reflex edge. Although there is no empty
circle that contains \( e \), the depicted circumsphere of \( e \) encloses no vertex that is visible from the
relative interior of \( e \). There are two vertices inside the circle, but
both are hidden behind segments. Hence, \( e \) is constrained Dela-
uay. Similarly, the circumsphere of \( t \) contains two vertices, but both
are hidden from the interior of \( t \) by segments, so \( t \) is constrained
Delaunay.

The main result of this paper is that if \( X \) contains a simplex \( s \),
then \( X \) must contain every lower-dimensional face of \( s \),
including its vertices. Any two simplices of a PLC, if one is not a
face of the other, may intersect only at a shared lower-dimensional
face or vertex.

Several definitions are needed prior to a formal statement of the
main result. Say that the visibility between two points \( p \) and \( q \) in
\( E^d \) is occluded if there is a \((d-1)\)-simplex \( s \) of \( X \) such that \( p \) and \( q \)
lie on opposite sides of the hyperplane that contains \( s \), and the line
segment \( pq \) intersects \( s \) (either in the boundary or in the relative
interior of \( s \)). If either \( p \) or \( q \) lies in the hyperplane containing \( s \),
then \( s \) does not occlude the visibility between them. Simplices in
\( X \) of dimension less than \( d-1 \) do not occlude visibility. The points
\( p \) and \( q \) can see each other if there is no occluding \((d-1)\)-simplex
of \( X \).

Let \( s \) be a \( k \)-simplex (for any \( k \)) whose vertices are in \( X \) (but
\( s \) is not necessarily a constraining simplex of \( X \)). Let \( S \) be a (full-
dimensional) sphere in \( E^d \); \( S \) is a circumsphere of \( s \) if \( S \) passes
through all the vertices of \( s \). If \( k = d \), then \( s \) has a unique cir-
cumsphere; otherwise, \( s \) has infinitely many circumspheres. The
simplex \( s \) is Delaunay if there is a circumsphere of \( S \) such that
no vertex of \( X \) lies inside \( S \). The simplex \( s \) is strongly Delaunay
if there is a circumsphere \( S \) of \( s \) such that no vertex of \( X \) lies inside
or on \( S \), except the vertices of \( s \). Every 0-simplex is trivially
strongly Delaunay.

The simplex \( s \) is constrained Delaunay if no constraining sim-
plex of \( X \) intersects the interior of \( s \) unless it contains \( s \) in its en-
tirety, and there is a circumsphere \( S \) of \( s \) such that no vertex of \( X \)
inside \( S \) is visible from any point in the relative interior of \( s \).

A PLC \( X \) is said to be ridge-protected if each constraining sim-
plex in \( X \) of dimension \( d-2 \) or less is strongly Delaunay.

The main result of this paper is that if \( X \) is ridge-protected,
and if no \( d+2 \) vertices of \( X \) lie on a common sphere, then the

![Figure 2: The edge \( e \) and triangle \( t \) are each constrained Delaunay. Bold lines represent segments.](image1)

![Figure 3: Schönhardt's untetrahedralizable polyhedron (right) is formed by rotating one end of a triangular prism (left), thereby creating three diagonal reflex edges.](image2)
constrained Delaunay $d$-simplices defined on the vertices of $X$ collectively form a triangulation of $X$. This triangulation is called the **constrained Delaunay triangulation** of $X$. The condition that $X$ be ridge-protected holds trivially in two dimensions (hence the success of two-dimensional CDTs), but not in three or more.

Note that it is not sufficient for each lower-dimensional constraining simplex to be Delaunay; if Schönhardt’s polyhedron is specified so that all six of its vertices lie on a common sphere, then all of its edges (and its faces as well) are Delaunay, but it still does not have a tetrahedralization. It is not possible to place the vertices of Schönhardt’s polyhedron so that all three of its reflex edges are strongly Delaunay (though any two may be).

If some $d - 2$ vertices of $X$ are cospherical, the existence of a CDT can be established through perturbation arguments (not presented here), but the CDT is not necessarily unique. There may be constrained Delaunay $d$-simplices whose interiors are not disjoint. Some of these simplices must be omitted to yield a proper triangulation.

It is appropriate, now, to consider a more general definition of PLC. To illustrate the limitation of the definition given heretofore, consider finding a tetrahedralization of a three-dimensional cube. If each square face is represented by two triangular constraining simplices, then the definition given above requires that the diagonal edge in each face be a constraining edge of $X$. However, the result of this paper may be obtained even if these edges are not strongly Delaunay. Hence, a PLC may contain constraining facets, which are more general than constraining simplices. Each facet is a polytope of any dimension from one to $d - 1$, possibly with holes and lower-dimensional facets inside it. A $k$-simplex $s$ is said to be a constraining simplex in $X$ if $s$ appears in the $k$-dimensional CDT of some $k$-facet in $X$. If $X$ is ridge-protected, then the CDT of a facet is just its Delaunay triangulation, because the boundary simplices of each facet are strongly Delaunay.

Figure 4 illustrates a three-dimensional PLC. As the figure illustrates, a facet may have any number of sides, may be nonconvex, and may have holes, slits, or vertices inside it. Two facets, if one is not a boundary facet of the other, may intersect only at shared lower-dimensional facets and vertices. See Miller et al. [8] for a complete list of restrictions.

The advantage of employing whole facets, rather than individual constraining simplices, is that only the lower-dimensional boundary facets of a $(d - 1)$-facet need be included in the PLC (including interior boundaries, at holes and slits). The lower-dimensional faces that are introduced in the relative interior of a $(d - 1)$-facet when it is triangulated do not need to be strongly Delaunay for this paper’s result to hold.

This advantage does not extend to lower-dimensional facets, because a ridge-protected $k$-facet, for $k \leq d - 2$, must be composed of strongly Delaunay $k$-simplices, and it is easy to show that any lower-dimensional face of a strongly Delaunay simplex is strongly Delaunay. Hence, in a ridge-protected PLC, all simplicial faces in the Delaunay triangulation of a facet are required to be strongly Delaunay, except for the faces in the triangulation of a $(d - 1)$-facet that are not in the facet’s boundary.

Testing whether a PLC is ridge-protected is straightforward. Form the Delaunay triangulation of the vertices of the PLC. If a constraining simplex $s$ is missing from the triangulation, then $s$ is not strongly Delaunay. Otherwise, $s$ is Delaunay; testing whether $s$ is strongly Delaunay is a local operation equivalent to determining whether the dual face of $s$ in the corresponding Voronoi diagram is nondegenerate.

Why is it useful to know that ridge-protected PLCs have CDTs? Although a given PLC $X$ may not be ridge-protected, it can be made ridge-protected by splitting simplices that are not strongly Delaunay into smaller simplices, with the insertion of additional vertices. The result is a new ridge-protected PLC $Y$, which has a CDT. The CDT of $Y$ is not a CDT of $X$, because it has vertices that $X$ lacks, but it is what I call a **conforming constrained Delaunay triangulation** (CCDT) of $X$: conforming because of the additional vertices, and constrained because its simplices are constrained Delaunay (rather than Delaunay).

One advantage of a CCDT over a conforming Delaunay triangulation is that the number of additional vertices needed is generally smaller. Once a PLC $X$ has been augmented to form a ridge-protected PLC $Y$, the Delaunay triangulation of the vertices of $Y$ contains all the constraining simplices of $Y$ of dimension $d - 2$ or smaller (because they are strongly Delaunay), but does not necessarily respect the $(d - 1)$-simplices of $Y$. The result of this paper implies that the CDT of $Y$ may be formed without adding any more vertices to $Y$. This idea stands in apposition to the most common method of recovering unrepresented facets in three-dimensional Delaunay-based mesh generation algorithms [14, 6, 15, 10], wherein additional vertices are inserted within the facet (for instance, where an edge of the tetrahedralization intersects the missing facet).

Because of the large disparity between the number of vertices required for two-dimensional constrained and conforming Delaunay triangulations, I suspect that asymptotically more vertices are needed (in the worst case) to produce a Delaunay tetrahedralization that conforms to a set of segments and facets in $E^3$ than are needed to produce a Delaunay tetrahedralization that merely conforms to the segments. Because the latter can be converted into a CCDT, the former may not be needed.

The utility of the CCDT is further buttressed by a practical algorithm for three-dimensional mesh generation that I describe elsewhere [13]. One variant of the Delaunay refinement algorithm described therein uses the constrained Delaunay property of the tetrahedra it produces to prove guaranteed bounds on the quality of the final mesh. The results of this paper are the necessary underpinnings of that algorithm.

### 2 Proof of Existence of the CDT

Throughout the proof, the terms “simplex” and “convex hull” refer to closed, convex sets of points; hence, they include all the points on their boundaries and in their interiors.

Let $X$ be a ridge-protected PLC in $E^d$, and suppose that some subset of $d + 1$ vertices in $X$ is affinely independent. The set of constrained Delaunay $d$-simplices defined on $X$ forms a triangulation of the vertices of $X$ if $X$ contains no $d + 2$ vertices that lie on a common sphere. This fact may be proven in two steps; first, by showing that every point in the convex hull of $X$ is contained in some constrained Delaunay $d$-simplex of $X$; second, by showing that any two constrained Delaunay $d$-simplices have disjoint interiors, and that the $d$-simplices are aligned on their $(d - 1)$-faces. Hence, the union of the constrained Delaunay $d$-simplices is the convex hull of $X$, and the constrained Delaunay $d$-simplices (and
their faces) form a simplicial complex. Only the second step requires the assumption that no \(d + 2\) vertices are cospherical.

Both steps are based on a straightforward procedure for “growing” a simplex, vertex by vertex. The procedure is presumed to be in possession of a constrained Delaunay \(k\)-simplex \(s_k\), whose vertices are \(v_0, \ldots, v_k\), and produces a constrained Delaunay \((k+1)\)-simplex \(s_{k+1}\) that possesses a face \(s_k\) and an additional vertex \(v_{k+1}\).

Because \(s_k\) is constrained Delaunay, it has a circumsphere \(S\) that encloses no vertex visible from the interior of \(s_k\). The growth procedure expands \(S\) like a bubble, so that its center \(O_S\) moves in a direction orthogonal to the \(k\)-dimensional hyperplane \(h_k\) that contains \(s_k\), as illustrated in Figure 5. Because \(O_S\) is moving in a direction orthogonal to \(h_k\), \(O_S\) remains equidistant from the vertices of \(s_k\), and hence it is always possible to choose a radius for \(S\) that ensures that \(S\) continues to pass through all the vertices of \(s_k\). The expansion ends when \(S\) contacts an additional vertex \(v_{k+1}\) that is visible from some point in the interior of \(s_k\). Once it is found, the forthcoming Theorem 5 guarantees that \(v_{k+1}\) is visible from every point in \(s_k\). The simplex \(s_{k+1}\) defined by \(v_{k+1}\) and the vertices of \(s_k\) is constrained Delaunay, a fact attested to by the circumsphere \(S\), which can be shown (again by Theorem 5) to enclose no vertex visible from any point in the interior of \(s_{k+1}\).

The motion of the sphere center \(O_S\) is governed by a direction vector \(c_k\), which is constrained to be orthogonal to \(h_k\), but may otherwise be specified freely. In the limit as \(O_S\) moves infinitely far away, \(S\) will approach the \((d - 1)\)-dimensional hyperplane that contains \(s_k\) and \(h_k\) and is orthogonal to \(c_k\). The region enclosed by \(S\) will approach an open half-space bounded by this hyperplane. A point is said to be above \(h_k\) if it lies in this open half-space. Any vertex outside \(S\) that \(S\) comes in contact with while expanding must lie above \(h_k\); the portion of \(S\) below \(h_k\) is shrinking toward the inside of \(S\).

A special case occurs if the sphere \(S\) already contacts a vertex \(u\) not in \(s_k\) before \(S\) begins expanding. If \(u\) is affinely independent of \(s_k\), and is visible from the interior of \(s_k\), then it is immediately accepted as \(v_{k+1}\). Otherwise, \(u\) is ignored. (The affinely independent case can only occur for \(k \geq 2\); for instance, when a triangle grows to become a tetrahedron, as illustrated in Figure 6, there may be ignored vertices on the triangle’s planar circumcircle.)

If the procedure is to succeed, \(c_k\) must satisfy the following visibility hypothesis. The open half-space above \(h_k\) must contain a vertex of \(X\) that is visible from the interior of \(s_k\). In the case of unconstrained Delaunay triangulations, this simply means that the half-space contains a vertex of \(X\). Theorem 4, to follow, clarifies the case of constrained Delaunay triangulations by showing that if \(X\) is ridge-protected and there is a vertex above \(h_k\), then there is a vertex above \(h_k\) that is visible from an arbitrarily chosen point in \(s_k\). Hence, for a ridge-protected PLC, the visibility hypothesis holds in the constrained case whenever it holds in the unconstrained case.

This growth procedure is the basis for the following proof.

**Theorem 1** Let \(p\) be any point in the convex hull of \(X\). If \(X\) is ridge-protected, then some constrained Delaunay \(d\)-simplex of \(X\) contains \(p\).

**Proof:** The proof is based on a two-stage constructive procedure similar to one used by Fortune [5] to prove the existence of unconstrained Delaunay triangulations. The first stage finds an arbitrary constrained Delaunay \(d\)-simplex, and involves \(d + 1\) steps, numbered zero through \(d\). For step zero, define a sphere \(S\) whose center is an arbitrary vertex \(v_0\) in \(X\), and whose radius is zero. The vertex \(v_0\) seeds the starting simplex \(s_0 = \{v_0\}\).

During the remaining steps, illustrated in Figure 7, \(S\) expands according to the growth procedure described above. At step \(k + 1\), the direction vector \(c_k\) is chosen so that some vertex of \(X\) lies above \(h_k\), and thus the visibility hypothesis is established. Such a choice of \(c_k\) is always possible because \(X\) contains \(d + 1\) affinely independent vertices, and thus some vertex is not in \(h_k\). Theorem 4 guarantees that some vertex is visible above \(h_k\) from the interior of \(s_k\). If there are several vertices visible, let \(v_{k+1}\) be the first such vertex contacted by the expanding sphere, so that no vertex inside the sphere is visible from the interior of \(s_k\). Theorem 5 shows that the new simplex \(s_{k+1}\) is constrained Delaunay.

After step \(d\), \(s_d\) is a constrained Delaunay \(d\)-simplex. If \(s_d\) contains \(p\), the procedure is finished; otherwise, the second stage begins.

Consider the directed line segment \(qp\), where \(q\) is an arbitrary point in the interior of \(s_d\) chosen so that \(qp\) does not intersect any simplices (defined on the vertices of \(X\)) of dimension \(d - 2\) or smaller, except possibly at the point \(p\). (Such a choice of \(q\) is always possible; the set of points in \(s_d\) that don’t satisfy this condition has measure zero.) The second stage “walks” along the segment \(qp\), traversing a sequence of constrained Delaunay \(d\)-simplices that intersect \(qp\) (illustrated in Figure 8), until it finds one that contains \(p\).

Wherever \(qp\) exits a \(d\)-simplex, the next \(d\)-simplex may be constructed as follows. Let \(s\) be the face through which \(qp\) exits the current \(d\)-simplex. The \((d - 1)\)-simplex \(s\) is either a constraining simplex in \(X\), or is constrained Delaunay. \(s\) is the base from which another constrained Delaunay \(d\)-simplex is grown, with the direction vector chosen to be orthogonal to \(s\) and directed out of the previous \(d\)-simplex. Is the visibility hypothesis satisfied? Clearly, the point \(p\) is above \(s\). Some vertex of \(X\) must lie above \(s\), because if none did, the convex hull of \(X\) would not intersect the half-space.
above \( s \) and thus would not contain \( p \). Hence, by Theorem 4, some vertex above \( s \) is visible from the interior of \( s \).

If \( s \) is constrained Delaunay, a new constrained Delaunay \( d \)-simplex may be formed as in the first stage. If \( s \) is a constraining \((d-1)\)-simplex in \( X \), but is not constrained Delaunay, then the growth procedure must be modified slightly to account for the fact that \( s \) does not have a circumsphere that encloses no vertex visible from \( s \). In this case, the “expanding” sphere \( S \) begins with its center infinitely far below \( s \) (so that the “inside” of \( S \) is the open half-space below \( s \)), and the portion of \( S \) above \( s \) is initially empty but expands as usual. A different proof (Theorem 8) is needed to show that the new \( d \)-simplex is constrained Delaunay.

Because the new \( d \)-simplex is above \( s \), it is distinct from the previous \( d \)-simplex. Because each successive \( d \)-simplex intersects a subsegment of \( q \) having nonzero length, each successive \((d-1)\)-face intersects \( q \) closer to \( p \), and thus no simplex is visited twice. Since only a finite number of simplices can be defined (over a finite set of vertices), the procedure must terminate; and since the procedure will not terminate unless the current \( d \)-simplex contains \( p \), there exists a Delaunay \( d \)-simplex that contains \( p \).

This procedure is recognizable as the basis for a well-known algorithm, called gift-wrapping, graph traversal, or incremental search, for constructing Delaunay triangulations [2]. Gift-wrapping begins by finding a single Delaunay \( d \)-simplex, which is used as a seed upon which the remaining Delaunay \( d \)-simplices crystallize one by one. Each \((d-1)\)-face of a Delaunay \( d \)-simplex is used as a base from which to search for the vertex that serves as the apex of an adjacent \( d \)-simplex. Theorem 1 shows that gift-wrapping can be used to produce CDTs as well, at least for ridge-protected PLCs.

As every point in the convex hull of \( X \) is contained in a constrained Delaunay \( d \)-simplex, it remains only to show that the set of constrained Delaunay \( d \)-simplices do not occupy common volume or fail to intersect neatly. It is only here that the assumption that no \( d+2 \) vertices are cospherical is needed.

**Theorem 2** Suppose that no \( d+2 \) vertices of \( X \) lie on a common sphere. Then the constrained Delaunay \( d \)-simplices of \( X \) (and their faces) collectively form a simplicial complex.

**Proof:** First, I show that constrained Delaunay \( d \)-simplices have disjoint interiors. Suppose for the sake of contradiction that some point \( p \) lies in the interior of two distinct constrained Delaunay \( d \)-simplices \( s \) and \( t \). Because \( s \) and \( t \) are constrained Delaunay, every vertex of \( s \) and \( t \) is visible from \( p \).

Let \( S_s \) and \( S_t \) be the circumspheres of \( s \) and \( t \). \( S_s \) and \( S_t \) cannot be identical, because \( s \) and \( t \) each have at least one vertex not shared by the other; if \( S_s \) and \( S_t \) were the same, at least \( d+2 \) vertices would lie on \( S_s \). \( S_s \) cannot enclose \( S_t \), nor the converse, because \( S_s \) and \( S_t \) enclose no vertices visible from \( p \). Hence, either \( S_s \) and \( S_t \) are entirely disjoint (and thus so are \( s \) and \( t \)), or their intersection is a \((d-1)\)-dimensional circle or point and is contained in a \((d-1)\)-dimensional hyperplane \( h \), as Figure 9 illustrates. (If \( S_s \) and \( S_t \) intersect at a single point, \( h \) is chosen to be tangent to both spheres.) Without loss of generality, suppose \( h \) is oriented horizontally, with the center of \( S_t \) directly above the center of \( S_s \). Because \( p \) lies in the interiors of \( s \) and \( t \), either some vertex of \( s \) lies below \( h \), or some vertex of \( t \) lies above \( h \). In the former case, there is a vertex of \( s \) inside \( S_t \) that is visible from a point \( p \) inside \( t \). So \( t \) is not constrained Delaunay. In the latter case, there is a vertex of \( t \) inside \( S_s \) that is visible from a point inside \( s \), so \( s \) is not constrained Delaunay. Either case implies a contradiction, so \( s \) and \( t \) have disjoint interiors and can intersect only at their boundaries.

Recall from the proof of Theorem 1 that if \( s \) is a constrained Delaunay \( d \)-simplex, then from any \((d-1)\)-face of \( s \) not in the
boundary of the convex hull of \( X \), one can find an adjoining constrained Delaunay \( d \)-simplex that shares that \((d-1)\)-face. No other constrained Delaunay \( d \)-simplex may occupy the same volume. It follows that constrained Delaunay \( d \)-simplices meet neatly on their \((d-1)\)-faces; and since the union of all constrained Delaunay \( d \)-simplices is the convex hull of \( X \), the constrained Delaunay \( d \)-simplices collectively form a simplicial complex.

It is now straightforward to show that every constraining simplex in \( X \) appears as a face of the CDT. Simply note that from any constraining simplex in \( X \), one may grow a constrained Delaunay \( d \)-simplex whose faces include the constraining simplex.

The remainder of this section is devoted to completing the proof of Theorem 1. The first step is to prove the visibility hypothesis. One potential difficulty is illustrated (for the three-dimensional case) in Figure 10. Imagine that you are standing on a \((d-1)\)-simplex, scanning the half-space above the simplex for a vertex that can serve as the apex of a \( d \)-simplex. Looking up into the sky, you see the three illustrated \((d-1)\)-facets, each of which occludes the apical vertex of another; the remaining vertices of these facets are hidden below the horizon (in the half-space below you). Hence, no vertex in the half-space is visible from your vantage point.

To prove the existence of a constrained Delaunay triangulation, one must show that this possibility is precluded if \( X \) is ridge-protected. In Figure 10, observe that the inner edges of the three facets form a cycle of overlapping simplices. The proof operates through nor encloses any other vertex. Hence, \( p \) lies strictly above \( h \); hence, every point in \( p \), lies in or above \( h \); and every point in \( t \) lies in or below \( h \). Every vertex of \( s \) or \( t \) that lies in \( h \) must belong to both \( s \) and \( t \); hence, every point of \( s \) or \( t \) that lies in \( h \) belongs to both. Recall that there exists a point \( p_2 \), of \( s \) and a point \( p_1 \) of \( t \) such that \( p_2 \not\in t \) and \( p_2 \) lies between \( p \) and \( p_1 \). The point \( p_2 \) must lie strictly above \( h \), and \( p_1 \) lies in or below \( h \), so \( p \) lies strictly above \( h \).

Consider Figure 11, which depicts the two-dimensional cross-section through \( E^d \) that passes through \( p \) and the centers \( O_s \) and \( O_t \) of the spheres \( S_s \) and \( S_t \). Let \( y_s \) be the signed height of \( O_s \) above \( h \); \( y_t \) is negative if \( O_t \) is below \( h \). Similarly, let \( y_t \) be the signed height of \( O_t \) above \( h \), and let \( t \) be the signed height of \( p \) above \( h \). Let \( r_s \) and \( r_t \) be the radii of \( S_s \) and \( S_t \), respectively. Let \( p' \) be the orthogonal projection of \( p \) onto \( h \) (in other words, \( pp' \) is orthogonal to \( h \)), and let \( O' \) be the orthogonal projection of \( O_s \) onto \( h \), which coincides with the orthogonal projection of \( O_t \) onto \( h \). Let \( t \) be any point of intersection of \( S_s \) and \( S_t \); and if the spheres do not intersect, let \( I = O' \). Observe that if \( S_s \) and \( S_t \) intersect, then \( r_s^2 + |O' I|^2 \); otherwise, \( r_s^2 < y_s^2 + |O' I|^2 \). From these relationships and the Pythagorean Theorem, we have

\[
\Psi_p(t) - \Psi_p(s) = [|p O_s|^2 - |p O_t|^2 - r_s^2 + r_t^2] \\
\geq [(y_s - t)^2 + (|p O_t|^2)] \\
-[(y_t - t)^2 + (|p' O_t|^2)] \\
-[(y_s^2 + |O' I|^2)] + [y_t^2 + |O' I|^2] \\
= 2l(y_s - y_t). 
\]

Because \( t \) and \( y_s - y_t \) are both positive, \( \Psi_p(t) > \Psi_p(s) \).

**Example:**

Let \( O_s \) and \( O_t \) be the center and radius of \( S_s \), respectively. Consider the function \( \Psi_p(s) = |p O_s|^2 - r_s^2 \) defined over the set of strongly Delaunay simplices relative to a vantage point \( p \). The proof stands on the fact that if \( s \) and \( t \) are strongly Delaunay simplices and \( s \) overlaps \( t \) from the viewpoint \( p \), then \( \Psi_p(s) < \Psi_p(t) \). Hence, the overlap relation among strongly Delaunay simplices defines a partial order, and no cycle of consecutively overlapping strongly Delaunay simplices is possible.

It remains only to show that \( \Psi_p(s) < \Psi_p(t) \) if \( s \) overlaps \( t \) from the viewpoint \( p \). Because \( s \) contains a point that \( t \) lacks (namely \( p_s \)), \( s \) must also possess a vertex that \( t \) lacks; hence, \( S_s \) (which passes through this vertex) and \( S_t \) (which does not) are distinct. If \( S_s \) and \( S_t \) intersect in a \((d-1)\)-dimensional circle, let \( h \) be the \((d-1)\)-dimensional hyperplane that passes through the circle of intersection (recall Figure 9). Otherwise, let \( h \) be the \((d-1)\)-dimensional hyperplane tangent to \( S_s \) at the point of \( S_s \) nearest \( S_t \).

Assume without loss of generality that \( h \) is oriented horizontally, with the center of \( S_s \) directly above the center of \( S_t \). Observe that \( S_s \) encloses any portion of \( S_t \) above \( h \), and \( S_t \) encloses any portion of \( S_s \) below \( h \). Because \( s \) and \( t \) are strongly Delaunay, every vertex of \( s \), and hence every point in \( s \), lies in or above \( h \); and every point in \( t \) lies in or below \( h \). Every vertex of \( s \) or \( t \) that lies in \( h \) must belong to both \( s \) and \( t \); hence, every point of \( s \) or \( t \) that lies in \( h \) belongs to both. Recall that there exists a point \( p_2 \), of \( s \) and a point \( p_1 \) of \( t \) such that \( p_2 \not\in t \) and \( p_2 \) lies between \( p \) and \( p_1 \). The point \( p_2 \) must lie strictly above \( h \), and \( p_1 \) lies in or below \( h \), so \( p \) lies strictly above \( h \).

Consider Figure 11, which depicts the two-dimensional cross-section through \( E^d \) that passes through \( p \) and the centers \( O_s \) and \( O_t \) of the spheres \( S_s \) and \( S_t \). Let \( y_s \) be the signed height of \( O_s \) above \( h \); \( y_t \) is negative if \( O_t \) is below \( h \). Similarly, let \( y_t \) be the signed height of \( O_t \) above \( h \), and let \( t \) be the signed height of \( p \) above \( h \). Let \( r_s \) and \( r_t \) be the radii of \( S_s \) and \( S_t \), respectively. Let \( p' \) be the orthogonal projection of \( p \) onto \( h \) (in other words, \( pp' \) is orthogonal to \( h \)), and let \( O' \) be the orthogonal projection of \( O_s \) onto \( h \), which coincides with the orthogonal projection of \( O_t \) onto \( h \). Let \( I \) be any point of intersection of \( S_s \) and \( S_t \); and if the spheres do not intersect, let \( I = O' \). Observe that if \( S_s \) and \( S_t \) intersect, then \( r_s^2 + |O' I|^2 \); otherwise, \( r_s^2 < y_s^2 + |O' I|^2 \). From these relationships and the Pythagorean Theorem, we have

\[
\Psi_p(t) - \Psi_p(s) = [|p O_s|^2 - |p O_t|^2 - r_s^2 + r_t^2] \\
\geq [(y_s - t)^2 + (|p O_t|^2)] \\
-[(y_t - t)^2 + (|p' O_t|^2)] \\
-[(y_s^2 + |O' I|^2)] + [y_t^2 + |O' I|^2] \\
= 2l(y_s - y_t). 
\]

Because \( t \) and \( y_s - y_t \) are both positive, \( \Psi_p(t) > \Psi_p(s) \).

**Figure 10:** Spherical projection of the half-space above your vantage point.

**Figure 11:** A two-dimensional cross-section of \( E^d \) that passes through \( p \), \( O_s \), and \( O_t \). The hyperplane \( h \) is orthogonal to \( O_s O_t \), and thus extends orthogonally out of the page. Note that \( p_s \) and \( p_t \) do not necessarily lie in this cross-section; they are depicted here as a reminder of why \( p \) must lie above \( h \).
Edelsbrunner [3] presents an acyclicity theorem that is nearly identical to Lemma 3. The proof given here is much simpler.

The impossibility of a cycle of overlapping strongly Delaunay simplices is the key to proving the visibility hypothesis for ridge-protected PLCs.

Theorem 4 (Visibility Hypothesis) Let \( h \) be a \((d-1)\)-dimensional hyperplane, and let \( p \) be a point in \( h \). Suppose there is at least one vertex of \( X \) in the open half-space above \( h \). If \( X \) is ridge-protected, then at least one vertex of \( X \) in the open half-space above \( h \) is visible from \( p \).

Proof: There is at least one vertex above \( h \). Either it is visible from \( p \) and the result follows, or there must be a \((d-1)\)-facet of \( X \) occluding its visibility. Hence, at least a portion of the boundary of at least one \((d-1)\)-facet lies above \( h \). It follows that some simplex \( e \) (of dimension \( d-2 \) or less) in the boundary of a \((d-1)\)-facet is at least partly visible from \( p \) in the half-space above \( h \). (The only possible alternative is a single facet blotting out the whole sky, which would only be possible if facets could be curved.) Let \( m \) be a point in \( e \) that is above \( h \) and visible from \( p \), as illustrated in Figure 13. Assume without loss of generality that \( m \) is the relative interior of \( e \) by choosing \( e \) to be of as low dimension as possible; for instance, if \( m \) lies in an edge of a tetrahedron in \( E^5 \), choose \( e \) to be the edge and not the tetrahedron. If \( e \) is a vertex the result follows, so assume \( e \) is of dimension at least one.

The proof proceeds by “walking” from \( m \) toward a vertex of \( e \), replacing \( e \) with any simplex that occludes its visibility from \( p \), and continuing the walk on the new simplex. This process is repeated until a vertex visible from \( p \) is found.

Begin by observing that because \( e \) is a simplex that contains \( m \), \( e \) must have at least one vertex \( v \) above \( h \). (The other vertices of \( e \) might lie on or below \( h \).) If \( v \) is visible from \( p \), the result follows. Otherwise, \( m \) is visible from \( p \) but \( v \) is not.

Let \( n \) be the point nearest \( m \) on the line segment \( mv \) that is not visible from \( p \). (In other words, \( n \) is the first occluded point encountered when walking from \( m \) to \( v \).) The segment \( mn \) must intersect some \((d-1)\)-facets of \( X \) at some point \( m' \). (If there are several \((d-1)\)-facets occluding the view from \( p \) to \( n \), consider only the facet that intersects \( pm \) closest to \( p \), so that \( m' \) is visible from \( p \).) Since \( n \) is the first occluded point on \( mv \), \( m' \) must lie in the boundary of the occluding facet. Generally, \( m' \) will lie in the interior of a \((d-2)\)-simplex bounding the facet, but in some cases \( m' \) may lie in a lower-dimensional boundary; let \( e' \) be the lowest-dimensional boundary simplex that contains \( m' \). If \( e' \) is a vertex, the proof is complete, so assume \( e' \) is not a vertex.

Because \( e \) and \( e' \) lie in boundaries of facets in \( X \), \( e \) and \( e' \) are strongly Delaunay. Clearly, \( e' \) overlaps \( e \) from the viewpoint \( p \).

Let \( v' \) be a vertex of \( e' \) that is above \( h \). If \( v' \) is visible from \( p \), the proof is complete. Otherwise, find another facet-bounding simplex \( e'' \) of dimension \( d-2 \) or less that overlaps \( e' \) from the viewpoint \( p \), and search for one of its vertices. Repeat the process until a vertex visible from \( p \) is found. Because \( X \) contains only a finite number of constraining simplices, and because Lemma 3 rules out the possibility of a cycle of consecutively overlapping strongly Delaunay simplices, the search must end.

The visibility hypothesis tells us that if the open half-space above a \( k \)-simplex \( s_k \) contains a vertex, then at least one candidate vertex \( u \) above \( s_k \) is visible from the interior of \( s_k \). Can we find a constrained Delaunay simplex \( s_{k+1} \) by taking the convex hull of \( s_k \) and \( u \)? There are two catches. First, the candidate vertex \( u \), although visible from at least one point inside \( s_k \), might not be visible from all points inside \( s_k \). Second, some vertex visible from inside \( s_{k+1} \) might prevent \( s_{k+1} \) from being constrained Delaunay. The following theorem shows that these problems do not arise in the case that matters: when a sphere circumscribing \( s_k \) encloses no vertex visible from the interior of \( s_k \).

Theorem 5 Let \( X \) be a ridge-protected PLC. Let \( s_k \) be a constrained Delaunay \( k \)-simplex, for some \( k < d \). Let \( u \) be a vertex of \( X \) that is affinely independent of \( s_k \) and is visible from some point \( p \) in the relative interior of \( s_k \). Suppose that there is a sphere \( S \) that passes through \( u \) and all the vertices of \( s_k \), and encloses no vertex of \( X \) that is visible from \( p \). Let \( s_{k+1} \) be the convex hull of \( s_k \) and \( u \). Then no constraining facet of \( X \) intersects the relative interior of \( s_{k+1} \); unless it contains \( s_{k+1} \). Furthermore, no vertex inside \( S \) is visible from any point in the relative interior of \( s_{k+1} \); hence, \( s_{k+1} \) is constrained Delaunay.

The proof requires the following two lemmata.

Lemma 6 Let \( S \) be a sphere, and let \( H_S \) be the convex hull of all the vertices of \( X \) that lie on or inside \( S \). Let \( t \) be a strongly Delaunay simplex. Suppose that some point of \( t \) lies between two points of \( H_S \) but not in \( t \). Then at least one vertex of \( t \) lies inside \( S \).

Proof: Suppose, for the sake of contradiction, that all vertices of \( t \) lie on or outside \( S \). Because \( t \) is strongly Delaunay, there is some
sphere $S_i$ that circumscribes $t$, but has no other vertices on or inside it. Because there are points of $H_S$ not in $t$, a portion of $S$ must lie outside $S_i$. By assumption, some point of $t$ falls between two points of $H_S$; this point lies inside $S$ and hence cannot be a vertex of $t$, so $t$ has at least two vertices. Because these vertices are on or outside $S_i$ and $S_i$ cannot be identical to $S$, a portion of $S_i$ must lie outside $S$. It follows that the spheres $S$ and $S_i$ intersect in a $(d - 1)$-dimensional circle.

Let $h$ be the $(d - 1)$-dimensional hyperplane that contains the intersection of $S$ and $S_i$, as illustrated in Figure 14. Without loss of generality, suppose $h$ is oriented horizontally with the center of $S$ directly above the center of $S_i$. The vertices of $t$ lie on $S_i$ but not inside $S$, so all points of $t$ must lie on or below $h$. Because $t$ is strongly Delaunay, every vertex of $H_S$ lies on or above $h$, and only those vertices of $H_S$ that are shared with $t$ may lie on $h$. Therefore, any point of $H_S$ not in $t$ lies strictly above $h$. Hence, no point of $t$ can lie between two points of $H_S$ not in $t$. The result follows by contradiction.

Lemma 7: Let $S$ be a sphere, and let $H_S$ be the convex hull of all the vertices of $X$ that lie on or inside $S$. Let $q$ and $z$ be two points in $H_S$ with the property that $q$ intersects a strongly Delaunay simplex $e$ that contains neither $q$ nor $z$. Let $m$ be the intersection point, and let $m'$ be any point in $H_S$ from which $m$ is visible. If $X$ is ridge-protected, then there is a vertex of $X$ inside $S$ that is visible from $p$.

Proof: See Figure 15. By assumption, $q$ and $z$ do not lie in $e$. Because the point $m$ of $e$ lies between two points that are in $H_S$ but not in $e$, Lemma 6 implies that at least one vertex $v$ of $e$ lies inside $S$. If $v$ is visible from $p$, the result follows. Otherwise, observe that $m$ and $v$ are both in $H_S$ and are both in $e$. The point $m$ is visible from $p$, but $v$ is not.

Let $n$ be the point nearest $m$ on the line segment $mn$ that cannot see $p$. The line segment $pn$ must intersect the boundary of some $(d - 1)$-facet of $X$ at some point $m'$ (if there are several $(d - 1)$-facets occluding the view from $p$ to $m$, consider only the facet that intersects $pn$ closest to $p$). Let $e'$ be the lowest-dimensional boundary simplex that contains $m'$. Because $e'$ lies in the boundary of a facet, $e'$ is strongly Delaunay. Clearly, $e'$ overlaps $e$ from the viewpoint $p$. Observe that $p$ and $n$ both lie in $H_S$, but they cannot lie in $e'$; if either of them did, the facet containing $e'$ would not obstruct the visibility between them. Again, Lemma 6 implies that at least one vertex $e'$ of $e'$ lies inside $S$. If $e'$ is visible from $p$, the result follows. Otherwise, find another simplex $e''$ that overlaps $e'$ from the viewpoint $p$ and continue.

The act of iterating in this manner yields a sequence of strongly Delaunay simplices, each overlapping the previous one. These iterations must terminate because the number of constraining simplices in $X$ is finite and, by Lemma 3, a cycle of consecutively overlapping strongly Delaunay simplices is not possible. Hence, the procedure will eventually yield a vertex visible from $p$.

Proof of Theorem 5: Consider the first claim. Suppose, for the sake of contradiction, that some $j$-facet $f$ (for any $j < d$) intersects the relative interior of $s_{k+1}$ but does not contain $s_{k+1}$. Let $s$ be a $j$-simplex of the Delaunay triangulation of $f$ that intersects the relative interior of $s_{k+1}$. Because $s$ is constrained Delaunay, it follows that if $s$ contains any point in the interior of $s_{k+1}$, then $s$ contains $s_{k+1}$ entirely. By assumption, $f$ does not contain $s_{k+1}$, so $s$ lacks at least one vertex of $s_{k+1}$. Hence, $s$ either does not contain $u$ or does not contain any point in the relative interior of $s_{k+1}$.

There exists a simplex $t$, which is $s$ or a facet of $s$, that intersects the relative interior of $s_{k+1}$ and contains neither $u$ nor any point in the relative interior of $s_{k+1}$. How do we know this? If $s$ contains $u$, then $s$ does not contain any point in the relative interior of $s_{k+1}$; let $t$ be the $(j - 1)$-face of $s$ that does not have $u$ for a vertex. Because $s$ intersects the relative interior of $s_{k+1}$, so must $t$. On the other hand, if $s$ contains $s_{k+1}$, then $s$ does not contain $u$; let $t$ be any $(j - 1)$-face of $s$ that lacks one vertex of $s_{k+1}$ and intersects the relative interior of $s_{k+1}$. (There must be at least one such face between $u$ and the relative interior of $s_{k+1}$.) Otherwise, let $t = s$. In each of these cases, $t$ contains neither $u$ nor any point in the relative interior of $s_{k+1}$.

Let $r$ be a point in the relative interior of $s_{k+1}$ such that $ru$ intersects $t$. Note that $t$ contains neither $r$ nor $u$. Let $q$ be the point nearest $p$ on the line segment $pr$ such that $qu$ intersects a constraining simplex or facet of dimension $d - 2$ or less that contains neither $q$ nor $u$, as Figure 16 illustrates. To see that such a choice of $q$ exists, imagine sliding $q$ along $pr$ (which can see $u$) toward $r$, stopping when the line segment $qu$ first intersects an appropriate simplex. If $t$ is a constraining simplex of dimension $d - 2$ or less, then $q$ will stop sliding no later than when it reaches $r$. Alternatively, if $t$ is not such a simplex, then $t$ must be a $(d - 1)$-facet (as illustrated), with $t$ lying within it. Because $p$ can see $u$, the line segment $pu$ does not intersect $f$. Hence, the 2-simplex $\triangle pqu$ must intersect a boundary of $f$, so $q$ will stop sliding no later than when $qu$ meets this boundary.

Let $m$ be the intersection point of $qu$ and the constraining simplex, as illustrated. The point $m$ is visible from $p$, because the relative interior of $\triangle pqu$ intersects no obstructing facet (otherwise, $q$ would have stopped earlier). Lemma 7 holds that there is a vertex of $X$ inside $S$ that is visible from $p$. This contradicts the assump-
Figure 16: The circumstance depicted here, wherein \( p \) can see \( u \) but \( r \) cannot, could not occur if the vertices of \( f \) were on or outside \( S \) and the edges of \( f \) were strongly Delaunay.

![Figure 16](image16.png)

Figure 17: The circumstance depicted here, wherein \( p \) cannot see \( v \) but \( r \) can, could not occur if the vertices of \( f \) were on or outside \( S \) and the edges of \( f \) were strongly Delaunay.

![Figure 17](image17.png)

Theorem 8 Let \( s \) be a constraining \((d-1)\)-simplex of \( X \). Let \( u \) be a vertex of \( X \) that is above \( s \) and visible from some point \( p \) in the interior of \( s \). Suppose that there is a sphere \( S \) that passes through \( u \) and all \( d \) vertices of \( s \), and encloses no vertex of \( X \) that is above \( s \) and visible from a point in the interior of \( s \). Let \( s_d \) be the convex hull of \( s \) and \( u \). Then no facet of \( X \) intersects the interior of \( s_d \), and \( s_d \) is constrained Delaunay.

Proof: The proof is similar to that of Theorem 5, with a few modifications to account for the fact that there may be vertices of \( X \) inside \( S \) below \( s \). Consider the first claim. Suppose, for the sake of contradiction, that some facet intersects the interior of \( s_d \).

Let \( p' \) be a point on the line segment \( pu \) that is extremely close to \( p \). Specifically, \( p' \) is such a small distance above \( s \) that \( p' \) cannot see any vertex of \( X \) below \( s \) (because they are all occluded by \( s \)). Furthermore, for any vertex \( w \) of \( X \) above \( s \), the line \( wp' \) passes through the interior of \( s \). Such a choice of \( p' \) is always possible because \( p \) is in the relative interior of \( s \). Additionally, \( p' \) is close enough to \( p \) that the convex hull of \( s \) and \( p' \) intersects no facet above \( s \). Such a choice of \( p' \) is always possible because no facet of \( X \) can intersect the interior of \( s \) (except the facet that contains \( s \)).

Because \( s \) is a Delaunay triangle of a ridge-protected facet, no vertex of \( X \) coplanar with \( s \) is inside \( S \). Hence, if there is a vertex inside \( S \) that is visible from \( p' \), it lies above \( s \). Observe that \( p' \) lies in \( H_s \) and can see \( u \).

By repeating the argument from the proof of Theorem 5, with \( p \) replaced by \( p' \), one may deduce that there is a vertex \( w \) of \( X \) inside \( S \) that is visible from \( p' \); \( w \) must be above \( s \). Because the line \( wp' \) passes through the relative interior of \( s \), \( w \) is visible from some point in the relative interior of \( s \), a contradiction. Hence, \( s_d \) the interior of \( s_d \).

Next, consider the claim that the \( d \)-simplex \( s_d \) formed from \( s \) and \( u \) is constrained Delaunay. Suppose, for the sake of contradiction, that there is some point \( r \) in the interior of \( s_{k+1} \) that can see a vertex \( v \) of \( X \) inside \( S \), as Figure 17 illustrates. By assumption, \( v \) is not visible from \( p \).

Let \( f \) be the \((d-1)\)-facet that obstructs the visibility of \( v \) from \( p \); if there are several such facets, choose \( f \) to be the facet that intersects \( pu \) closest to \( p \).

Let \( q_v \) be the point nearest \( p \) on the line segment \( pr \) such that \( q_v \) intersects the boundary of a \((d-1)\)-facet that contains neither \( q \) nor \( v \), and the intersection point \( m_v \) is visible from \( p \). To see that such a choice of \( q \) exists, imagine sliding \( q \) along \( pr \) (whose view of \( v \) is obstructed by \( f \)) toward \( r \) (from which \( v \) is visible). As \( q \) moves, either \( q_v \) will strike the boundary of a \((d-1)\)-facet lying between \( q \) and \( f \), or the intersection of \( qv \) and \( f \) will remain visible from \( p \) until \( q_v \) intersects a boundary of \( f \) (which will occur before \( q \) reaches \( r \)).

Lemma 7 holds that there is a vertex of \( X \) inside \( S \) that is visible from \( p \). By contradiction, \( s_{k+1} \) is constrained Delaunay.

Theorem 5 shows that a constrained Delaunay simplex can be "grown" into a higher-dimensional constrained Delaunay simplex. However, a separate result is needed to show that a constraining \((d-1)\)-simplex of \( X \)—which might not be constrained Delaunay—can also be "grown" into a constrained Delaunay \( d \)-simplex.

3 Algorithms for Constructing CD Ts

Starting with one constrained Delaunay \( d \)-simplex, a naïve implementation of the foregoing gift-wrapping algorithm might construct each additional \( d \)-simplex by considering all of the vertices as candidates and testing the visibility of each one against every facet. The running time is thus \( O(d^2 n_n, n_d) \), where \( n_d \) is the number of \( d \)-simplices in the output, \( n_v \) is the number of input vertices, and \( n_f \) is the number of \((d-1)\)-simplices in the triangulations of the input \((d-1)\)-facets. (The factor of \( d^2 \) is the cost of testing whether a \((d-1)\)-simplex obstructs the visibility between two points.) This leaves much room for improvement, which I hope will be filled by future research. For instance, the sophisticated search algorithm and analysis techniques applied by Dwyer [2] to unconstrained gift-wrapping might be generalizable to the constrained case as well.
Here, I offer a few practical suggestions without asymptotic guarantees.

For most applications, the fastest way to form the CDT of a ridge-protected PLC (albeit not in the worst case) is to use the best available algorithm to find an unconstrained Delaunay triangulation of the input vertices, then recover the \((d - 1)\)-facets one by one. Each \((d - 1)\)-facet may be recovered by deleting the \(d\)-simplices whose interiors it intersects, then retriangulating the polytopes now left empty on each side of \(f\). It is easy to show that all of the simplices not thus deleted are still constrained Delaunay. Since a CDT of the new configuration exists, each empty polytope can be triangulated with constrained Delaunay simplices. If these polytopes are typically small, the performance of the algorithm used to triangulate them is not critical, and gift-wrapping will suffice.

Vertices may be incrementally inserted into and deleted from a CDT just like an ordinary Delaunay triangulation, so long as the underlying PLC (which changes incrementally with the triangulation) remains ridge-protected. When a vertex is inserted, the simplices that are no longer constrained Delaunay are deleted. When a vertex is deleted, the simplices that contain it are deleted. In either case, the resulting polytopal hole is retriangulated to complete the new CDT. As with facet recovery, the existence of a CDT of the entire underlying PLC ensures that a CDT of the hole can be produced. Hence, the best approach to triangulating a PLC might be to start with a Delaunay triangulation of the vertices of the \((d - 1)\)-facets, then recover the \((d - 1)\)-facets themselves, and then finally insert the remaining vertices incrementally.

The ability to incrementally insert and delete vertices is also useful for mesh generation, especially in circumstances where the constrained Delaunay property can be used to establish provable properties of the meshing algorithm [13].

Unfortunately, subsets of \(d + 2\) or more cospherical vertices can cause real difficulties for gift-wrapping. A gift-wrapping algorithm may make decisions that are mutually inconsistent, and find itself unable to complete the triangulation. For an example affecting unconstrained Delaunay tetrahedralizations in \(E^5\), imagine a large vertex set that includes six cospherical vertices whose surroundings have been inadvertently tetrahedralized so as to form a hollow space shaped like Schönhardt’s polyhedron. This problem can be solved by using symbolic perturbation to simulate general position, thereby ensuring that all decisions made by gift-wrapping are mutually consistent. Additionally, cospherical or nearly-cospherical vertices create the need for exact arithmetic when performing the insphere tests associated with Delaunay triangulation.

4 Conclusions

In their paper on two-dimensional conforming Delaunay triangulations, Edelsbrunner and Tan [4] write:

A seemingly difficult open problem is the generalization of our polynomial bound to three dimensions. The somewhat easier version of the generalized problem considers a graph whose vertices are embedded as points in \(\mathbb{R}^3\), and edges are represented by straight line segments connecting embedded vertices. More relevant, however, is the problem for the crossing-free embedding of a complex consisting of vertices, edges, and triangles.

The present result shifts the emphasis back to the former of these two problems. An algorithm that could create a CCDT by inserting only a modest number of additional vertices on input segments might have great practical importance.

Several other questions also deserve investigation. Are there higher-dimensional constrained Delaunay triangulation algorithms that have the same running time as optimal algorithms for unconstrained Delaunay triangulations? Do higher-dimensional CDTs have optimality properties such as minimizing the largest min-containment sphere, as higher-dimensional Delaunay triangulations do [9]? Is there a less conservative definition of “constrained Delaunay” (perhaps allowing visibility to be affected by constraining simplices of dimension less than \(d - 1\)) that defines triangulations over a larger class of PLCs?

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References