"IMPLICIT FAIRING OF IRREGULAR MESHES USING DIFFUSION AND CURVATURE FLOW"

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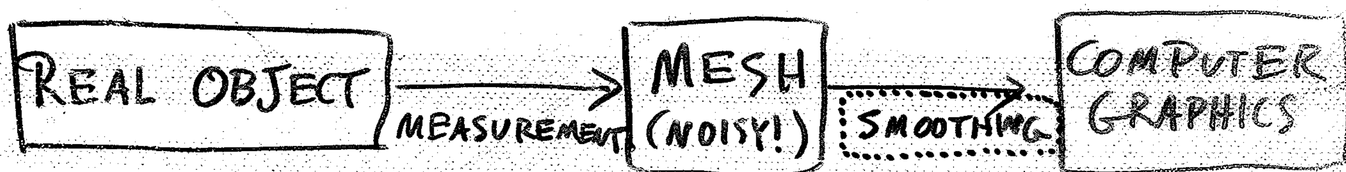
PROBLEM

GIVEN: TRIANGULAR SURFACE MESH T (VERTICES $v_1, \ldots, v_N$, EDGES e_{ij})

GOAL: PRODUCE A NEW MESH T' WITH LARGE-SCALE GEOMETRIC FEATURES OF T, BUT WITHOUT THE NOISE. (T' IS A SMOOTHER T)

ASSUMPTION: WE REALLY DO WANT A SMOOTH SURFACE

APPLICATION



SOLUTION

GENERAL APPROACH

- SMOOTHING OPERATOR $F_\alpha : \{\text{MESHES}\} \to \{\text{MESHES}\}$
  α = AMOUNT OF SMOOTHING ($\alpha > 0$)

$T_0 \leftarrow T$
$k \leftarrow 0$
Do:
  $T_{k+1} \leftarrow F_{\alpha_k}(T_k)$
  $k \leftarrow k+1$
UNTIL T_k IS SMOOTH ENOUGH (OR FOR SOME PREDETERMINED NUMBER OF STEPS)
$T' \leftarrow T_k$
RETURN T'

QUESTIONS

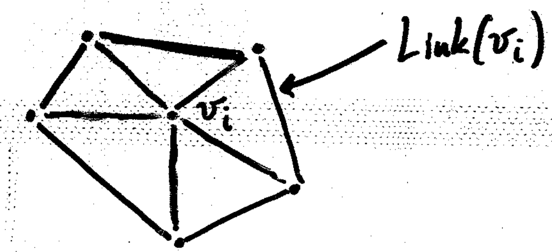
1) HOW TO CHOOSE GOOD F_α? IT SHOULD:
   a) BE "ACCURATE" (NOT DISTORT GEOMETRY, AND REPRESENT OUR INTUITION FOR SMOOTHNESS)
   b) BE EFFICIENTLY IMPLEMENTABLE
2) HOW MUCH TO SMOOTH IN ONE STEP? LARGE STEPS ARE GOOD, BUT STEP SIZE IS LIMITED BY THE STABILITY OF F_α.

F_α IS STABLE IF WE CAN TAKE BIG STEPS WITHOUT THE MESH GOING CRAZY.

EASY STARTING POINT: LAPLACIAN SMOOTHING

NOTATION

RECALL - **LINK** OF A VERTEX v_i (Link(i))



CONVENTION: #Link(i) = NUMBER OF VERTICES ADJACENT TO v_i

DISCRETE LAPLACIAN

$$L(v_i) = \frac{1}{\#\mathrm{Link}(i)} \sum_{j \in \mathrm{Link}(i)} (v_j - v_i)$$

LAPLACIAN SMOOTHING

$$F_\alpha(T) = T + \alpha L(T)$$

(VIEW T AS A VECTOR WITH 3N COMPONENTS; $L(T)$ IS L APPLIED TO EACH VERTEX)

ADVANTAGES: EASY, FAST, LINEAR

DISADVANTAGES:
- DISTORTS GEOMETRY
- PULLS VERTICES AWAY FROM DETAILED REGIONS
- ONLY STABLE IF $\alpha < 1$

SOME PERSPECTIVE:

$L(v_i)$ APPROXIMATES THE CONTINUOUS LAPLACIAN:

$$\Delta \vec{x} = \frac{\partial^2 \vec{x}}{\partial u^2} + \frac{\partial^2 \vec{x}}{\partial v^2} \qquad \left(\begin{array}{l}\text{SURFACE PARAMETRIZED} \\ \text{BY } \vec{x}(u,v)\end{array}\right)$$

... AND ITERATING

$$T_{k+1} = T_k + \alpha L(T_k)$$

IS LIKE SOLVING

$$\frac{dT}{dt} = \lambda \Delta T$$

DIFFUSION PROCESS (VERY "STIFF")

BY EULER'S METHOD USING FINITE DIFFERENCES.

DIFFUSION MAKES SENSE: SMALL BUMPS/DENTS DISSIPATE RAPIDLY. <u>HOWEVER</u>:

1) STIFF PROBLEMS ARE NOT AMENABLE TO EXPLICIT METHODS (STABILITY ISSUES).

2) DISCRETE LAPLACIAN AS DEFINED IS FAULTY AS AN APPROXIMATION TO Δ.

BOTH PROBLEMS CAN BE FIXED.

BIG IDEA

FIX FOR STABILITY PROBLEM: REPLACE EULER'S METHOD WITH A STABLE IMPLICIT METHOD, E.G., IMPLICIT EULER.

$$\underline{T_{k+1}} = T_k + \alpha L(\underline{T_{k+1}})$$ IMPLICIT EQUATION TO SOLVE FOR T_{k+1}

SINCE L IS LINEAR:

$$T_{k+1} - \alpha L(T_{k+1}) = T_k$$
$$(I - \alpha L)T_{k+1} = T_k$$ $3N \times 3N$ SYSTEM (N = # OF VERTICES)

BIG SYSTEM, BUT SPARSE. USE A SPARSE SOLVER LIKE PRECONDITIONED BICONJUGATE GRADIENTS (PBCG)

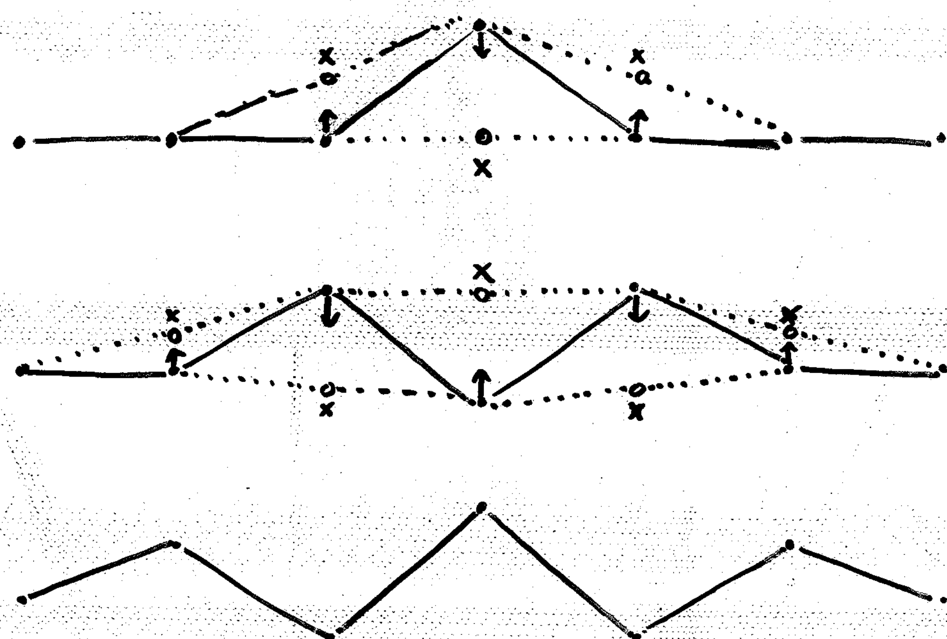
IMPLICIT EULER IS B-STABLE
$\Rightarrow$ ARBITRARILY LARGE STEPS WILL STILL MOVE IN THE RIGHT DIRECTION

IN PRACTICE: BENEFIT OF TAKING LARGE STEPS MORE THAN OFFSETS COST OF SOLVING THE SYSTEM IF HIGH DEGREE OF SMOOTHING IS SOUGHT.

INSTABILITY OF EXPLICIT SMOOTHING

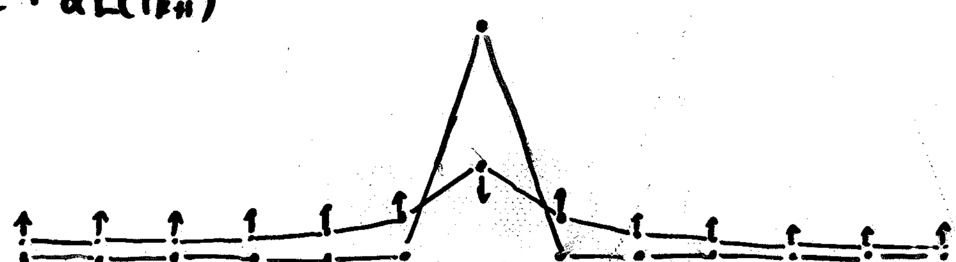
5.5

E.G. $T_{k+1} = T_k + \alpha L(T_k)$, $\alpha \geq 1$



WITH IMPLICIT SMOOTHING:

$T_{k+1} = T_k + \alpha L(T_{k+1})$



(NOTE: ALL THE VERTICES MOVE)

FIX FOR FAULTY APPROXIMATION TO Δ: SCALE-DEPENDENT DISCRETE LAPLACIAN

$$\tilde{L}(v_i) = 2\,\frac{\displaystyle\sum_{j \in \mathrm{Link}(i)} \frac{v_j - v_i}{|v_j - v_i|}}{\displaystyle\sum_{j \in \mathrm{Link}(i)} |v_j - v_i|}$$

ADVANTAGE: DEALS WITH NONUNIFORM MESHES MUCH BETTER THAN DISCRETE LAPLACIAN (MOSTLY RESPECTS GEOMETRY)

DISADVANTAGES:

- VERTICES STILL SLIDE AWAY FROM REGIONS OF CONCENTRATION.
- $\tilde{L}(v_i)$ IS NONLINEAR AS A FUNCTION OF VERTEX POSITIONS. (BUT WE CAN LINEARIZE IT BY PRETENDING THE EDGE LENGTHS $|v_j - v_i|$ DON'T CHANGE DURING A SOLUTION STEP. WORKS IN PRACTICE!)

TO FIX VERTEX SLIDING: NEED SMOOTHING OPERATOR THAT MOVES VERTICES ONLY IN **NORMAL** DIRECTION...

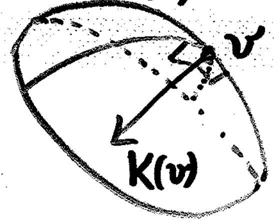
CURVATURE FLOW

IDEA: MOVE VERTICES PROPORTIONALLY TO <u>MEAN CURVATURE NORMAL</u>.

<u>WHY</u>: MEAN CURVATURE NORMAL IS "GRADIENT" OF LOCAL SURFACE AREA NEGATED, SO FLOWING TOWARD IT SHOULD MINIMIZE SURFACE AREA $\Rightarrow$ SMOOTH THE MESH.

<u>HOW:</u>

<u>CONTINUOUS</u>: $$K(v) = \frac{-\nabla \text{Area}(S(v))}{2\,\text{Area}(S(v))}$$

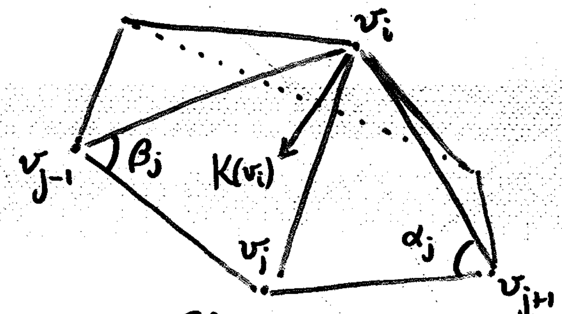


$K(v)$ = MEAN CURV. NORMAL

$S(v)$ = "INFINITESIMAL" PATCH OF SURFACE CONTAINING v, BOUNDARY FIXED

$\nabla \text{Area}(S(v))$ IS GRADIENT W.R.T. COORDINATES OF v

<u>DISCRETE</u>: $$K(v_i) = \frac{-\nabla \text{Area}(\text{Star}(v_i))}{2\,\text{Area}(\text{Star}(v_i))}$$



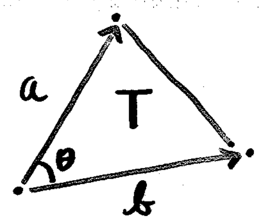
$\text{Star}(v_i)$

$$= \frac{\sum_{j \in \text{Link}(i)} (\cot\alpha_j + \cot\beta_j)(v_j - v_i)}{4\,\text{Area}(\text{Star}(v_i))}$$

SMOOTHING STEP: $T_{K+1} = T_K + \alpha K(T_{K+1})$

$\rightarrow$ VERTICES NO LONGER SLIDE!

COMPUTATIONAL REMARKS

1. 

$$\text{Area}(T) = \frac{1}{2}\sqrt{|a|^2|b|^2 - (a\cdot b)^2}$$

$$\cot(\theta) = \frac{a\cdot b}{2\,\text{Area}(T)}$$

2. DEAL WITH NONLINEARITY OF K AS BEFORE.

FURTHER ISSUES

1. MESH WILL SHRINK. SOLUTION: RESCALE AFTER EACH SOLUTION STEP TO PRESERVE ENCLOSED VOLUME. (VOLUME FORMULA DIRECTLY GENERALIZES POLYGON AREA; SEE PROF. SHEWCHUK'S LECTURE NOTES ON GEOMETRIC ROBUSTNESS.)

2. TO FIX A VERTEX: IMPOSE $L(v_i)=0$. OTHER TYPES OF CONSTRAINTS POSSIBLE.

3. HIGHER POWERS OF THE LAPLACIAN ($\Delta^2 = \Delta\circ\Delta$, $\Delta^3 = \Delta\circ\Delta\circ\Delta$, ...) THEORETICALLY SMOOTH BETTER, BUT THE SYSTEM BECOMES LESS SPARSE. GOOD TRADEOFF: Δ^2.