Vector Space of Circles

`Topics:` weighted distance and orthogonality, lifting, combination, coaxal systems.

Weighed distance and orthogonality. Everything we say in this section applies equally well to circles and to spheres in any fixed dimension. It is therefore convenient to limit the language to circles in the plane. Recall the definition of weighted distance of a point x from a circle $\hat{p}$ with center p and radius P:

$$\pi_{\hat{p}}(x) \quad = \quad \|x-p\|^2 - P^2.$$

Let this weighted distance be X^2. If we draw a circle with radius X around x, as we do in figure 6, we intersect the circle $\hat{p}$ at a right angle. This suggests we extend the notion of weighted distance to pairs of circles:

$$\pi(\hat{p},\hat{x}) \quad = \quad \|x-p\|^2 - P^2 - X^2.$$

The two circles are *orthogonal*, denoted as $\hat{p} \perp \hat{x}$, if $\pi(\hat{p},\hat{x}) = 0$. Orthogonal circles enjoy interesting properties, some of which are best studied from a three-dimensional perspective.

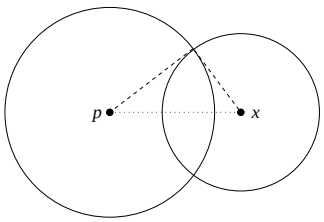

Figure 6: The square distance between p and x is the sum of square radii.

Lifting. For a circle $\hat{p} = (p, P^2)$ we consider the lifted plane and its polar point:

$$\begin{aligned} f \quad = \quad f_{\hat{p}}(x) \quad &= \quad \|x-p\|^2 - P^2 - \|x\|^2, \\ g \quad = \quad g_{\hat{p}} \quad &= \quad (p, -\|p\|^2 + P^2). \end{aligned}$$

It is easy to visualize the plane and the point in $\mathbb{R}^3$. The plane intersects the paraboloid $\varpi(x) = -\|x\|^2$ in an ellipse whose projection to the plane is the circle. The point g lies above the paraboloid such that the patch visible from g is exactly the part of the paraboloid above the plane. The ellipse forms the silhouette of the paraboloid as seen from g, and the rest of the paraboloid is invisible from g. This is illustrated in one lower dimension in Figure 7. The lifted image of orthogonality

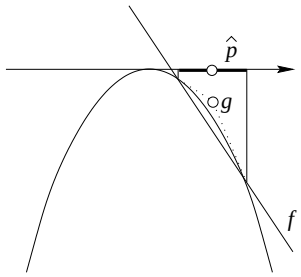

Figure 7: An interval $\hat{p}$, its lifted line f, and the polar point g.

is incidence. To explain this we transform the algebraic expression of the former into that of the latter:

$$\begin{aligned} \hat{p} \perp \hat{x} \quad &\text{iff} \quad \|p-x\|^2 - P^2 - X^2 \quad = \quad 0 \\ &\text{iff} \quad \|p-x\|^2 - P^2 - \|x\|^2 \quad = \quad -\|x\|^2 + X^2 \\ &\text{iff} \quad g_{\hat{x}} \in f_{\hat{p}}. \end{aligned}$$

We state this result for later reference.

Claim 3. $\hat{p} \perp \hat{x}$ iff $g_{\hat{x}} \in f_{\hat{p}}$.

Combination. We are interested in generating new circles from old ones. It is easier to do this in $\mathbb{R}^3$, where we have points and we can add and scale using the vector space structure of $\mathbb{R}^3$. Note first that we have a one-to-one correspondence between the points in $\mathbb{R}^3$ and the circles in $\mathbb{R}^2$:

$$(p, P^2) \quad \longleftrightarrow \quad (p, -\|p\|^2 + P^2).$$

The position of the point relative to the paraboloid is

$$\left.\begin{array}{l}\text{above}\\ \text{on}\\ \text{below}\end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} P^2 > 0,\\ P^2 = 0,\\ P^2 < 0.\end{array}\right.$$

In the third case, the square radius P^2 is negative and the radius P is a positive multiple of the imaginary unit. Given two circles $\hat{p}$ and $\hat{q}$ we can now define what we mean by adding and scaling. The *sum*, $\hat{p}+\hat{q}$, is the circle whose corresponding point is $g_{\hat{p}} + g_{\hat{q}}$, and the *scaling*, $\alpha \cdot \hat{p}$, is the circle whose corresponding point is $\alpha \cdot g_{\hat{p}}$. It is quite easy to obtain an intuitive understanding for scaled sums that are affine combinations of circles. Recall that the affine hull of two circles is

$$\text{aff}\,\{\hat{p},\hat{q}\} \quad = \quad \{\hat{r} = \alpha \cdot \hat{p} + \beta \cdot \hat{q} \mid \alpha + \beta = 1\}.$$

In $\mathbb{R}^3$, it corresponds to the line passing through the points $g_{\hat{p}}$ and $g_{\hat{q}}$, and also the pencil of planes passing through the line $f_{\hat{p}} \cap f_{\hat{q}}$. The latter picture immediately implies that all circles in the affine hull of $\hat{p}$ and $\hat{q}$ pass through the intersection points of $\hat{p}$ and $\hat{q}$, whether real or imaginary. More generally, they all share the same line of equal weighted distance.

Claim 4. If $\pi_{\hat{p}}(x) = \pi_{\hat{q}}(x) = c$ then $\pi_{\hat{r}}(x) = c$ for all $\hat{r} \in \text{aff}\,\{\hat{p},\hat{q}\}$.

Coaxal systems. Here is another way to think of Claim 4. If x has equal distance from $\hat{p}$ and $\hat{q}$ then we can choose a radius X so that $\hat{x} = (x, X^2)$ is orthogonal to both $\hat{p}$ and $\hat{q}$. By Claim 4, $\hat{x}$ is also orthogonal to all $\hat{r}$ in the affine hull of $\hat{p}$ and $\hat{q}$. In the same way we construct another circle $\hat{y} = (y, Y^2)$ orthogonal to all $\hat{r}$. Now take the affine hull of $\hat{x}$ and $\hat{y}$. Every $\hat{r}$ is orthogonal to $\hat{x}$ and $\hat{y}$ and therefore also to every $\hat{z} \in \text{aff}\,\{\hat{x},\hat{y}\}$. We just constructed a coaxal system of circles: every circle in the first affine hull is orthogonal to every circle in the second affine hull, see Figure 8. How can we interpret a coaxal system in $\mathbb{R}^3$? Think of the affine hull of $\hat{p}$ and $\hat{q}$ as a pencil of planes. Then $\hat{x}$ and $\hat{y}$ are points on the line common to all planes, and the affine hull of $\hat{x}$ and $\hat{y}$ is that line.

Coaxal systems can be generalized to three and higher dimensions. For every $1 \le i \le d+1$ and $j = d+2-i$ there are $i+j$ $(d-1)$-dimensional spheres in $\mathbb{R}^d$ such that each of the j spheres is orthogonal to each of the i spheres. We take affine hulls separately and get an $(i-1)$-flat F_1 and a $(j-1)$-flat F_2 of spheres. Every sphere in F_1 is orthogonal to every sphere in F_2. In the plane we have three cases, namely $i/j = 1/3,\ 2/2,\ 3/1$. The only non-trivial case is $2/2$, which is explained above. In three-dimensional space we have four cases, namely $i/j = 1/4,\ 2/3,\ 3/2,\ 4/1$. The first and the last are trivial, and the middle two are symmetric.

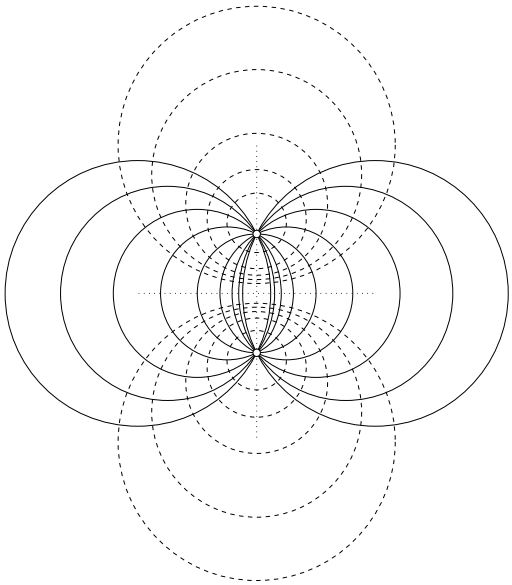

Figure 8: Coaxal system of orthogonal circles.

Bibliographic notes. The idea of mimicking the vector space of points in $\mathbb{R}^3$ for the purpose of manipulating circles in $\mathbb{R}^2$ goes back to independent work by Clifford, Darboux, and Frobenius in the middle of the nineteenth century, see the survey article [2]. Dan Pedoe uses the idea prominently in his textbook in geometry [1].

[1] D. Pedoe. *Geometry. A Comprehensive Course.* Dover, New York, 1988.

[2] R. E. Pfiefer and C. van Hook. Circles, vectors, and linear algebra. *Math. Mag.* **66** (1993), 75–86.