Non-convex Polyhedra

Topics: polyhedra, faces, inclusion-exclusion formula, example, boolean formula.

Polyhedra. Recall from Section 19 that a general and therefore not necessarily convex *polyhedron* is a finite union of convex polyhedra:

$$P = \bigcup_{i \in \Lambda} \bigcap H_i,$$

where Λ is a finite index set and each H_i is a finite set of closed half-spaces. As a running example for this section we use the polyhedron in Figure 5. It can be described as the union of four rectangular boxes.

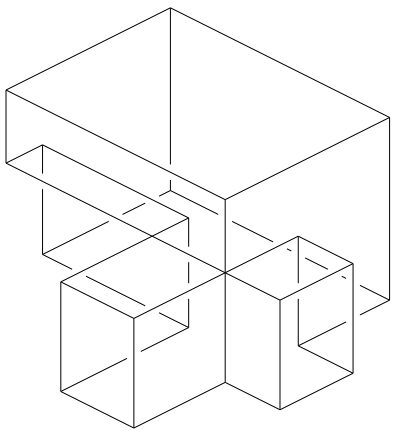

Figure 5: A polyhedron with various types of vertex neighborhoods.

Faces. Just like in three dimensions we base the definition of a face on the idea of face figures. Let b be the open ball with unit radius centered at the origin in $\mathbb{R}^d$. For a point $x \in \mathbb{R}^d$ we choose $\varepsilon > 0$ sufficiently small and intersect the open ball with radius ε around x with the polyhedron: $N_\varepsilon(x) = (x + \varepsilon \cdot b) \cap P$. The *face figure* of x is the enlarged version of this neighborhood:

$$\mathrm{ff}\, x = x + \bigcup_{\lambda > 0} \lambda \cdot (N_\varepsilon(x) - x).$$

In the generic case, a vertex belongs to d hyperplanes. In $\mathbb{R}^3$ this is three planes, and there are only six different types of face figures, all shown in Figure 6. As an

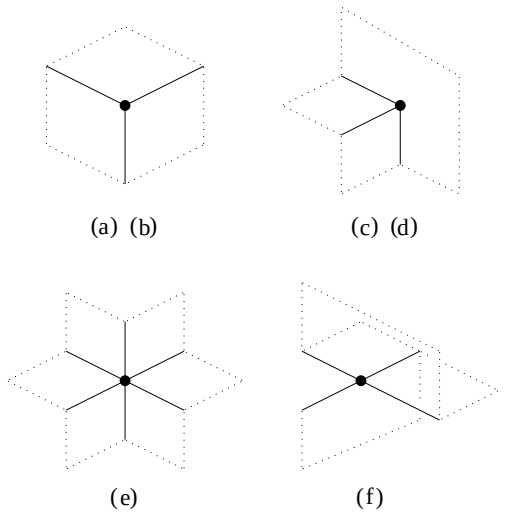

Figure 6: The six types of generic vertex figures in three-dimensional space. The figures (e) and (f) are symmetric from both sides.

exercise identify the different types of vertex figures in Figure 5. A *face* F is the closure of a maximal set of points with identical face figure, and the *face figure* of F is the common face figure of the points in the set before taking the closure. This definition allows for faces that consist of more than one component. In Figure 5 there is one segment (1-face) that consists of two components. Which segment is that?

Inclusion-exclusion formula. In order to generalize Theorem 2A from convex to non-convex polyhedra, we rewrite the alternating sum in terms of faces. For the time being, let $P = \bigcap H$ be a d-dimensional convex polyhedron and Φ be the set of non-empty faces. Each face $F \in \Phi$ is defined by a collection of half-spaces $X \subseteq$

H, that is,

$$F \;=\; F_X \;=\; P \cap \bigcap_{h \in X} \operatorname{bd} h.$$

In the generic case, the size of X is the codimension of F, which is $\operatorname{codim} F = d - \dim F$. Now we can rewrite Theorem 2A as

$$\sum_{F \in \Phi} (-1)^{\operatorname{codim} F} \cdot \gamma_F(y) \;=\; \begin{cases} 1 & \text{if } x \in \bigcap H, \\ 0 & \text{if } y \notin \bigcap H, \end{cases}$$

where $\gamma_F(y) = \gamma_X(y)$ if $F = F_X$. Recall that $\gamma_F(y) = 1$ iff y is outside every half-space in X. Define the *negative face figure* of F as the reflection of the face figure through a point x that satisfies $\operatorname{ff} x = \operatorname{ff} F$:

$$\operatorname{nff} F \;=\; 2x - \operatorname{ff} F.$$

With this notation we can interpret γ_F as the indicator function of the negative face figure: $\gamma_F(y) = 1$ iff $y \in \operatorname{nff} F$. For technical reasons we restrict our attention to polyhedra P that have the *manifold property*. By this we mean that P is a manifold with boundary and so is every face of P. Now we can state the generalization of Theorem 2A.

THEOREM 2C. If P is a d-dimensional polyhedron that is bounded and has the manifold property then

$$(-1)^d \cdot \sum_{F \in \Phi} \chi_F \cdot \gamma_F(y) \;=\; \begin{cases} 1 & \text{if } x \in \bigcap H, \\ 0 & \text{if } y \notin \bigcap H. \end{cases}$$

The proof is too complicated to be presented here. Note that Theorem 2C agrees with Theorem 2A if P is convex. Indeed, $\chi_F = (-1)^{\dim F}$ for every face F of the convex polyhedron, and

$$(-1)^d \cdot \chi_F \;=\; (-1)^{d + \dim F} \;=\; (-1)^{\operatorname{codim} F}.$$

Example. We check Theorem 2C for the polyhedron in Figure 5. Figure 7 redraws that polyhedron with edges and vertices solid if the viewpoint y (your eye) lies in their negative face figures. The number of facets, edges, vertices with y in the negative face figure are 10, 15, 6. We also count the polyhedron since $\operatorname{nff} P = \mathbb{R}^3$ and get

$$-1 + 10 - 15 + 6 \;=\; 0,$$

which is correct because y is outside P.

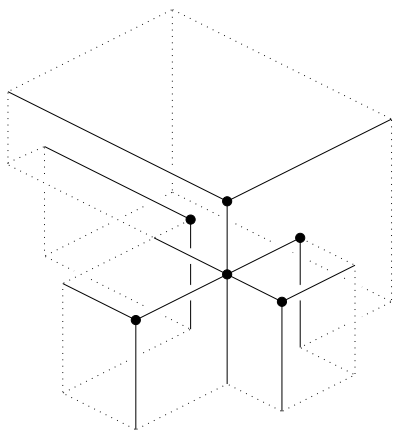

Figure 7: The solid edges and vertices have the viewpoint (your eye) in the negative face figure.

Boolean formula. The sum in Theorem 2C evaluates to 0 or to 1 but never to any other integers, so we might as well restrict our attention to the parity of its terms. Specifically, let $\Phi' \subseteq \Phi$ be the set of faces with odd Euler characteristic. Theorem 2C implies that the sum of parities of the $\gamma_F(y)$ over all $F \in \Phi'$ is odd if $y \in P$ and even if $y \notin P$. The parity of the sum of two bits is the exclusive-or operation:

$$\begin{aligned} 1 + 1 &= 0, \\ 1 + 0 &= 1, \\ 0 + 1 &= 1, \\ 0 + 0 &= 0. \end{aligned}$$

The boolean exclusive-or corresponds to the set theoretic symmetric difference. We can thus reinterpret the above parity result as saying that y belongs to the symmetric difference of all negative face figures iff y lies inside the polyhedron:

$$P \;=\; \bigoplus_{F \in \Phi'} \operatorname{nff} F.$$

We can further decompose each negative face figure into a symmetric difference of intersections of open half-spaces. In the generic case each intersection has at most d half-spaces. If we consider d a constant then the number of terms in the formula is a constant times the number of faces.

Bibliographic notes. The material for this section is taken from Edelsbrunner [2]. The right way to define faces of a polyhedron is a debated issue in the mathematics literature, see [3], and the definition in this section is taken from [2]. However, [2] defines the Euler

characteristic of a face as the alternating sum of Betti numbers, so his main result seems different from Theorem 2C, on the surface, but can readily be translated into our form by compensating for the opposite sign in every odd dimension. The problem of finding short boolean formulas for polyhedra was raised by Peterson [4] and by and large solved in the two-dimensional case by Dobkin et al. [1]. The symmetric difference formulation of Theorem 2C is the first general answer to Peterson's question in dimension three and higher.

[1] D. P. Dobkin, L. J. Guibas, J. E. Hershberger and J. Snoeyink. An efficient algorithm for finding the CSG representation of a simple polygon. *Algorithmica* **10** (1993), 1–23.

[2] H. Edelsbrunner. Algebraic decomposition of non-convex polyhedra. *In* "Proc. 36th Ann. IEEE Sympos. Found. Comput. Sci. 1995", 248–257.

[3] B. Grünbaum and G. C. Shephard. A new look at Euler's theorem for polyhedra. *Amer. Math. Monthly* **101** (1994), 109–128.

[4] D. Peterson. Halfspace representations of extrusions, solids of revolution, and pyramids. Rept. SAND84-0572, Sandia National Laboratories, Albuquerque, New Mexico, 1984.