Error Measure

`Topics:` signed distance, fundamental quadric, error, eigenvalues and eigenvectors.

Signed distance. The surface simplification algorithm measures the error of an edge contraction as the sum of square distances of a point from a collection of planes. A plane with unit normal vector v_i and offset δ_i contains all points p with orthogonal projection $\delta_i \cdot v_i$:

$$h_i = \{p \in \mathbb{R}^3 \mid p^T \cdot v_i = -\delta_i\},$$

see Figure 1. The signed distance of a point $x \in \mathbb{R}^3$ from the plane h_i is

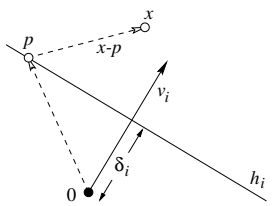

Figure 1: We use the unit normal vector to define the signed distance from h_i so v_i points from the negative to the positive side.

$$\begin{aligned} d(x, h_i) &= (x - p)^T \cdot v_i \\ &= x^T \cdot v_i + \delta_i \\ &= \mathbf{x}^T \cdot \mathbf{v}_i, \end{aligned}$$

where $\mathbf{x}^T = (x^T, 1)$ and $\mathbf{v}_i{}^T = (v_i{}^T, \delta_i)$. In words, the signed distance in $\mathbb{R}^3$ can be expressed as a scalar product in $\mathbb{R}^4$ as illustrated in Figure 2.

Fundamental quadric. The sum of square distances of a point x from a collection of planes H is

$$\begin{aligned} E_H(x) &= \sum_{h_i \in H} d^2(x, h_i) \\ &= \sum_{h_i \in H} (\mathbf{x}^T \cdot \mathbf{v}_i) \cdot (\mathbf{v}_i{}^T \cdot \mathbf{x}) \\ &= \mathbf{x}^T \cdot (\sum_{h_i \in H} \mathbf{v}_i \cdot \mathbf{v}_i{}^T) \cdot \mathbf{x}, \end{aligned}$$

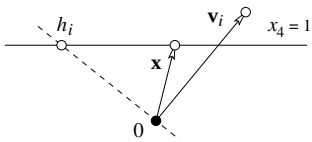

Figure 2: The 3-dimensional space $x_4 = 1$ is represented by the horizontal line. It contains point x and plane h_i, which in the 1-dimensional representation are both points.

where

$$\mathbf{Q} = \sum \mathbf{v}_i \cdot \mathbf{v}_i{}^T = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}.$$

is a symmetric 4-by-4 matrix referred to as the *fundamental quadric* of the map $E_H : \mathbb{R}^3 \to \mathbb{R}$. The sum of square distances is non-negative, so $\mathbf{Q}$ is positive semi-definite. The error of an edge contraction is obtained from an error function like $E = E_H$. Let $\mathbf{x}^T = (x_1, x_2, x_3, 1)$ and note that

$$\begin{aligned} E(x) &= \mathbf{x}^T \cdot \mathbf{Q} \cdot \mathbf{x} \\ &= Ax_1^2 + Ex_2^2 + Hx_3^2 \\ &\quad + 2(Bx_1x_2 + Cx_1x_3 + Fx_2x_3) \\ &\quad + 2(Dx_1 + Gx_2 + Ix_3) \\ &\quad + J. \end{aligned}$$

We see that E is a quadratic map that is non-negative and unbounded. Its graph can only be an elliptic paraboloid as illustrated in Figure 3. In other words, the preimage of a constant error value ϵ, $E^{-1}(\epsilon)$, is an ellipsoid. Degenerate ellipsoids are possible, such as cylinders with elliptic cross-sections and pairs of planes.

Error. The *error* of the edge contraction $ab \to c$ is the minimum value of $E(x) = E_H(x)$ over all $x \in$

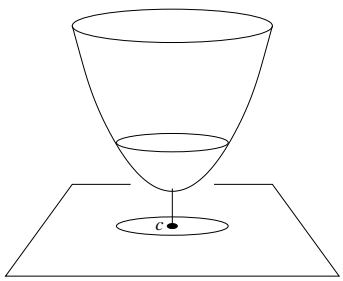

Figure 3: Illustration of $E = E_H$ in one lower dimension. The cross-section at a fixed height ϵ is an ellipse.

$\mathbb{R}^3$, where H is the set of planes spanned by triangles in the preimage of the star of the new vertex c. The geometric location of c is the point x that minimizes E. In the non-degenerate case this point is unique and can be computed by setting the gradient $\nabla E = (\partial E/\partial x_1, \partial E/\partial x_2, \partial E/\partial x_3)$ to zero. The derivative with respect to x_i is

$$\begin{aligned}\frac{\partial E}{\partial x_i}(x) &= \frac{\partial \mathbf{x}^T}{\partial x_i} \cdot \mathbf{Q} \cdot \mathbf{x} + \mathbf{x}^T \cdot \mathbf{Q} \cdot \frac{\partial \mathbf{x}}{\partial x_i} \\ &= \mathbf{Q}_i^T \cdot \mathbf{x} + \mathbf{x}^T \cdot \mathbf{Q}_i,\end{aligned}$$

where $\mathbf{Q}_i^T$ is the i-th row of $\mathbf{Q}$. The point $c \in \mathbb{R}^3$ that minimizes $E(x)$ is the solution to the system of three linear equations $Q \cdot x = q$, where

$$Q = \begin{pmatrix} A & B & C \\ B & E & F \\ C & F & H \end{pmatrix} \text{ and } q = \begin{pmatrix} D \\ G \\ I \end{pmatrix}.$$

Hence $c = Q^{-1} \cdot q$, and the sum of square distances of c from the planes in H is $E(c)$. The equation for c sheds light on the possible degeneracies. The non-degenerate case corresponds to $\operatorname{rank} Q = 3$, the case of a cylinder corresponds to $\operatorname{rank} Q = 2$, and the case of two parallel planes corresponds to $\operatorname{rank} Q = 1$. Rank 0 is not possible because Q is the non-empty sum of products of unit vectors.

Eigenvalues and eigenvectors. We may translate the planes by $-c$ so E attains its minimum at the origin. In this case $D = G = I = 0$ and $J = E(0)$. The shape of the ellipsoid $E^{-1}(\epsilon)$ can be described by the eigenvalues and eigenvectors of Q. By definition, the *eigenvectors* are non-zero vectors x that satisfy $Q \cdot x = \lambda \cdot x$. The value of λ is the corresponding *eigenvalue*. The eigenvalues are the roots of the *characteristic polynomial* of Q, which is

$$\begin{aligned}P(\lambda) &= \det \begin{pmatrix} A-\lambda & B & C \\ B & E-\lambda & F \\ C & F & H-\lambda \end{pmatrix} \\ &= \det Q - \lambda \cdot \operatorname{dtr} Q + \lambda^2 \cdot \operatorname{tr} Q - \lambda^3,\end{aligned}$$

where $\det Q$ is the determinant, $\operatorname{dtr} Q$ is the sum of cofactors of the three diagonal elements, and $\operatorname{tr} Q$ is the trace of Q. For symmetric positive semi-definite matrices the characteristic polynomial has three non-negative roots, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. Once we have an eigenvalue we can compute the corresponding eigenvector to span the nullspace of the underconstrained system $(Q - \lambda) \cdot x = 0$.

What is the geometric meaning of eigenvectors and eigenvalues? For symmetric matrices the eigenvectors are pairwise orthogonal and can be viewed as defining another coordinate system for $\mathbb{R}^3$. The three symmetry planes of the ellipsoid $E^{-1}(\epsilon)$ coincide with the coordinate planes of this new system, see Figure 4. We can write the error function as

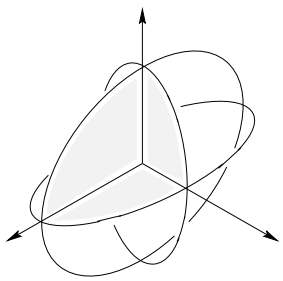

Figure 4: The ellipsoid is indicated by drawing the elliptic cross-sections along the three symmetry planes spanned by the eigenvectors.

$$\begin{aligned}E(x) &= \mathbf{x}^T \cdot \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & J \end{pmatrix} \cdot \mathbf{x} \\ &= \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^3 + J.\end{aligned}$$

The preimage for a fixed error $\epsilon > J$ is the ellipsoid with axes of half-lengths $\sqrt{(\epsilon - J)/\lambda_i}$ for $i = 1, 2, 3$.

Bibliographic notes. The idea of using the sum of square distances from face planes for surface simplification is due to Garland and Heckbert [1]. Eigenvalues and eigenvectors of matrices are topics in linear algebra. A very readable introductory text is the book by Gilbert Strang [2].

[1] M. Garland and P. S. Heckbert. Surface simplification using quadratic error metrics. *Computer Graphics*, Proc. siggraph 1997, 209–216.

[2] G. Strang. *Introduction to Linear Algebra.* Wellesley-Cambridge Press, Wellesley, Massachusetts, 1993.