Meeting 8 February 11, 1999

Euler Characteristic

`Topics:` alternating sums, shelling, shellability, cell complexes, 2-manifolds.

Alternating sums. The *Euler characteristic* of a simplicial complex K is the alternating sum of the number of simplices:

$$\chi(K) = s_0 - s_1 + s_2 - \ldots + (-1)^d s_d,$$

where $d = \dim K$ and s_i is the number of i-simplices in K. It is common to omit the (-1)-simplex from the sum. A simplex of even dimension contributes 1 and a simplex of odd dimension contributes -1, therefore

$$\chi(K) = \sum_{\emptyset \neq \sigma \in K} (-1)^{\dim \sigma}.$$

As an example consider the complex $B = \mathrm{Cl}\,\sigma$, which consists of σ and all its faces. Assuming $d = \dim \sigma$ we have

$$\begin{aligned} \chi(B) &= \sum_{i=0}^{d} (-1)^i \binom{d+1}{i+1} \\ &= (1-1)^{d+1} - (-1)^{-1} \binom{d+1}{0} \\ &= 1. \end{aligned}$$

The boundary complex of B is $S = B - \{\sigma\}$, and its Euler characteristic is

$$\begin{aligned} \chi(S) &= \chi(B) - (-1)^{\dim \sigma} \\ &= 1 - (-1)^{\dim \sigma}. \end{aligned}$$

This is 0 if the dimension of σ is even and 2 if the dimension is odd.

Shelling. Consider the case where K is a triangulation of the closed disk, $\mathbb{B}^2$. We draw K in the plane and get a triangulated polygon like the one shown in Figure 12. Our goal is to prove $\chi(K) = 1$. Since K can be any triangulation of $\mathbb{B}^2$, this amounts to proving that *every* triangulation of the closed disk has Euler characteristic equal to 1. We use the concept of shelling to prove this claim.

A *shelling* of K is an ordering of the triangles such that each prefix defines a triangulation of $\mathbb{B}^2$. K is *shellable* if it has a shelling. A shelling constructs K without every creating pinch points, disconnected pieces, or holes along the way, see Figure 12. Suppose

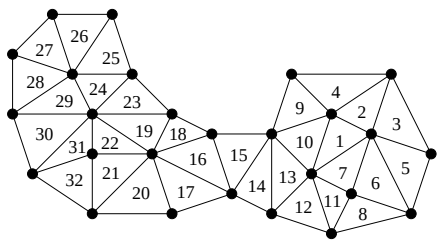

Figure 12: The numbers specify a shelling of the triangulation.

$\tau_1, \tau_2, \ldots, \tau_n$ is a shelling. When we add τ_i, for $i \geq 2$, its intersection with the complex of its predecessors either consists of a single edge with both vertices, or of two edges with three vertices, see Cases (b) and (c) in Figure 13. Case (a) occurs only for τ_1. Cases (d) through (m) cannot occur in a shelling because the union of either the first $i-1$ or the first i triangles in not homeomorphic to $\mathbb{B}^2$.

Let K_i be the closure of the set of first i triangles in the shelling. At the beginning we have $\chi(K_1) = 1$. If τ_i shares one edge with its predecessors then we effectively add one triangle, two edges, and one vertex. If τ_i shares two edges we add one triangle and one edge. In either case we have $\chi(K_i) = \chi(K_{i-1})$. In words, the Euler characteristic remains the same during the entire construction, and therefore $\chi(K) = \chi(K_n) = 1$. Modulo the existence of a shelling we have thus proved that *every triangulation* of the closed disk has Euler characteristic equal to 1. Although we cannot prove it, we state a vast generalization of this result.

THEOREM 2. If K and L are triangulations of the same topological space then $\chi(K) = \chi(L)$.

This result is the reason the Euler characteristic can be considered a property of the topological space, rather than of a triangulation of that space.

Shellability. To complete the proof that $\chi(K) = 1$ for every triangulation K of the closed disk we still need to find a shelling.

LEMMA 3. Every triangulation of $\mathbb{B}^2$ is shellable.

PROOF. We construct a shelling backwards. Specifically, we prove that every triangulation K of $n \geq 2$ triangles has a triangle τ_n that meets the rest in one of the two allowable configurations shown as Cases (b) and (c) in Figure 13.

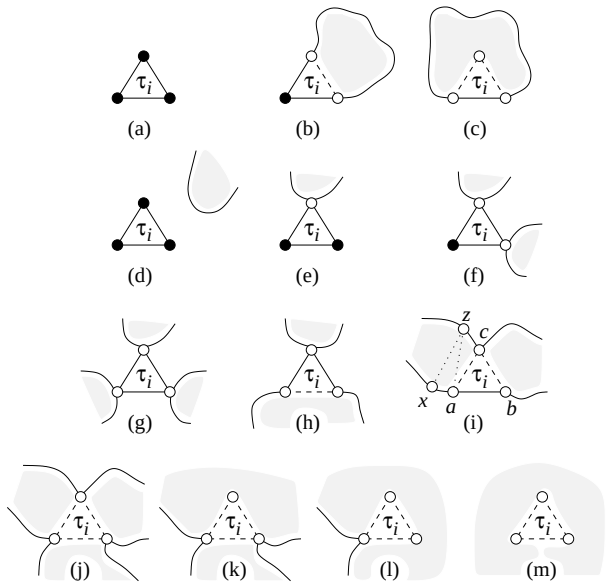

Figure 13: The 13 ways a triangle can intersect with the complex of its predecessors. Only cases (a), (b), (c) occur in a shelling.

`Case 1.` K contains no interior vertex. We consider the dual graph, whose nodes are the triangles in K and whose arcs connect two nodes if they share a common edge. The dual graph is a tree and therefore contains at least two leaves. A leaf triangle is connected to only one other node, which it meets in one edge as shown in Case (b) of Figure 13.

`Case 2.` K contains no leaf. Then K contains at least one interior vertex. Let ab be an edge on the boundary and abc the triangle next to ab. Either abc intersects the rest of K as in Case (c) of Figure 13, or c is also a boundary vertex, as in Case (i) of Figure 13. We may assume the latter. Let ax and zc be the first and last edge on the boundary path from a to c that does not pass through b. We are done if one of the two triangles next to ax and zc satisfies the requirements for τ_n. Otherwise, both triangles meet the rest as in Case (i), and we continue the search by substituting xz for ac. The search cannot continue indefinitely because K has only finite size. Eventually, we find a triangle τ_n that meets the rest in two edges as shown in Case (c) of Figure 13.

By induction we may assume that the complex defined by the remaining $n-1$ triangles has a shelling. We append τ_n and have a shelling of K. □

Cell complexes. The Euler characteristic in not restricted to simplicial complexes and can be computed as the alternating sum of the number of cells in more general complexes. If two complexes have homeomorphic underlying spaces we require they have the same Euler characteristic. For a large class of complexes this is indeed the case. Without exploring the most general setting that satisfies this requirement we consider finite *cell complexes*. Its cells are pairwise disjoint open balls, the boundary of a cell is the closure minus the cell, and the boundary of each cell is the union of other cells in the complex. An example is the dunce cap constructed from a triangular piece of paper by gluing the three edges as indicated in Figure 14.

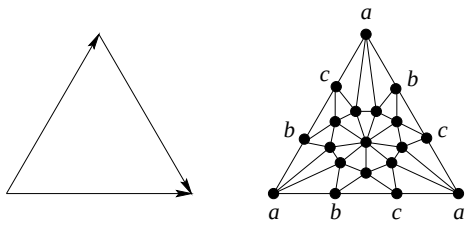

Figure 14: The dunce cap to the left consists of one 2-cell, one edge, and one vertex. Its triangulation to the right consists of 27 triangles, 39 edges, and 13 vertices.

Let Z be a finite cell complex. For technical reasons we assume there is a simplicial complex K and a homeomorphism $f : |Z| \to \|K\|$ such that the image of the closure of each cell is the underlying space of a subcomplex of K. K triangulates $|Z|$ as well as the closure of every cell in $z \in Z$. Let $S \subseteq B \subseteq K$ be subcomplexes that triangulate the boundary and the closure of z. We think of $B - S$ as a triangulation of the interior, which is

the cell itself. Since gluing along the boundary does not affect the interior we can pretend that S triangulates a sphere. The Euler characteristic of B_S is therefore

$$\begin{aligned}\chi(B-S) &= \chi(B)-\chi(S)\\ &= 1-(1-(-1)^d)\\ &= (-1)^d,\end{aligned}$$

where d is the dimension of z. We see that the contribution of $B-S$ to $\chi(K)$ is the same as the contribution of z to $\chi(Z)$. It follows that the Euler characteristics are the same: $\chi(Z)=\chi(K)$. This observation greatly simplifies the hand-computation of the Euler characteristic for many spaces.

2-manifolds. A 2-dimensional manifold can be constructed from a piece of paper by gluing edges along its boundary. As an example consider the torus, $\mathbb{T}$, which can be constructed from a square by gluing edges in opposite pairs as shown in Figure 15. The square together

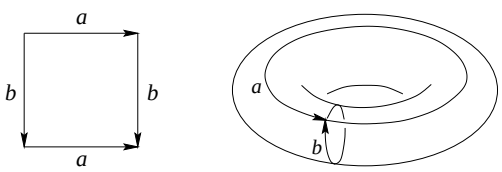

Figure 15: Edges with the same label are glued so their arrows agree. After gluing we have two edges and one vertex.

with its two edges and one vertex forms a cell complex for the torus, with Euler characteristic

$$\begin{aligned}\chi(\mathbb{T}) &= 1-2+1\\ &= 0.\end{aligned}$$

The straightforward treatment of the torus can be extended to general 2-manifolds using the complete characterization of 2-manifolds, which was one of the major achievements in nineteenth century mathematics. The list of orientable 2-manifolds consists of the 2-sphere, the torus with one hole, the torus with two holes, etc. The number of holes is the *genus* of the 2-manifold. The torus with g holes can be constructed from its polygonal schema, which is a regular $4g$-gon with edges

$$a_1a_2a_1^-a_2^-a_3a_4a_3^-a_4^- \ldots a_{2g-1}a_{2g}a_{2g-1}^-a_{2g}^-,$$

where an edge without minus sign is directed in counterclockwise and one with minus is directed in clockwise order around the $4g$-gon, see Figure 16. The g-hole

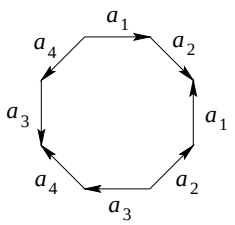

Figure 16: The polygonal schema of the double torus.

torus is constructed by gluing edges in pairs as indicated by the labels. After gluing we are left with $2g$ edges and one vertex. The Euler characteristic is therefore

$$\chi(\mathbb{T}_g) = 2-2g.$$

Given a triangulated orientable 2-manifold we can therefore use the Euler characteristic to compute the genus and decide the topological type of the 2-manifold.

Bibliographic notes. The history of the Euler characteristic is described in an entertaining book by Imre Lakatos, who studies progress in mathematics from a philosophical perspective [3]. The final word on the subject is contained in a seminal paper by Henri Poincaré, who proves that the Euler characteristic is equal to the alternating sum of Betti numbers [4]. Betti numbers are ranks of homology groups and will be discussed later in this course. The Euler characteristic has been developed and applied in a number of directions, see e.g. [5]. An algorithm for constructing a shelling of a triangulation of the closed disk is described in a paper by Danaraj and Klee [1]. A good treatment of the classification of 2-manifolds and their polygonal schemas can be found in the text by Peter Giblin [2].

[1] G. Danaraj and V. Klee. A representation of 2-dimensional pseudomanifolds and its use in the design of a linear time shelling algorithm. *Ann. Discrete Math.* **2** (1978), 53–63.

[2] P. J. Giblin. *Graphs, Surfaces and Homology.* Second edition, Chapman and Hall, London, 1981.

[3] I. Lakatos. *Proofs and Refutations.* Cambridge Univ. Press, Cambridge, England, 1976.

[4] H. Poincaré. Complément à l'analysis situs. *Rendiconti del Circolo Matematico di Palermo* **13** (1899), 285–343.

[5] Yu. A. Shashkin. *The Euler Characteristic.* Little Math. Library, Mir Publ., Moscow, 1989.