Spaces and Manifolds

Topics: topological spaces, homeomorphisms, triangulations, manifolds, manifolds with boundary, orientability.

Topological spaces. The treatment of topological ideas in these notes is mostly combinatorial in nature. To understand the connection to continuous phenomena we need some basic concepts from point set topology. The most fundamental of these is the notion of a *topological space*, which is a point set $\mathbb{X}$ together with a system X of subsets $A \subseteq \mathbb{X}$ that satisfy

(i) $\emptyset, \mathbb{X} \in X$,
(ii) $Z \subseteq X$ implies $\bigcup Z \in X$, and
(iii) finite $Z \subseteq X$ implies $\bigcap Z \in X$.

The system X is the *topology* of $\mathbb{X}$ and its sets are the *open sets* of $\mathbb{X}$. This definition is exceedingly general and therefore non-intuitive, but with time we will get a better feeling for what a topological space really is. The most important example for us is the *d-dimensional Euclidean space*, denoted as $\mathbb{R}^d$. We use the Euclidean distance function to define an *open ball* as the set of all points closer than some given distance from a given point. The topology of $\mathbb{R}^d$ is the system of open sets, where each open set is a union of open balls.

All other topological spaces in these notes are subsets of $\mathbb{R}^d$. A *topological subspace* of the pair $\mathbb{X}, X$ is a subset $\mathbb{Y} \subseteq \mathbb{X}$ together with the *subspace topology* consisting of all intersections between $\mathbb{Y}$ and open sets of $\mathbb{X}$: $Y = \{\mathbb{Y} \cap A \mid A \in X\}$. An example of a topological subspace of $\mathbb{R}^d$ is the *d-ball* whose points are at most unit distance from the origin,

$$\mathbb{B}^d = \{x \in \mathbb{R}^d \mid \|x\| \leq 1\}.$$

Its open sets are the intersections of $\mathbb{B}^d$ with open sets in $\mathbb{R}^d$. Note that an open set of the d-ball is not necessarily an open set of the Euclidean space.

Homeomorphisms. Topological spaces are considered the same or of the same type if they are connected the same way. What it means to be connected the same way still needs to be defined, and there are several possibilities. We use a function from one topological space to another. This function is *continuous* if the preimage of every open set is open, and if it is continuous it is referred to as a *map*. A *homeomorphism* is a function

$$f : \mathbb{X} \to \mathbb{Y}$$

that is bijective, continuous, and has a continuous inverse. If a homeomorphism exists then $\mathbb{X}$ and $\mathbb{Y}$ are *homeomorphic*, and this is denoted as $\mathbb{X} \approx \mathbb{Y}$. If we want to stress that $\approx$ is an equivalence relation we say that $\mathbb{X}$ and $\mathbb{Y}$ are *topologically equivalent* or that they have the same *topological type*. Figure 7 shows five examples of 1-dimensional spaces with pairwise different topological types. For another example consider the

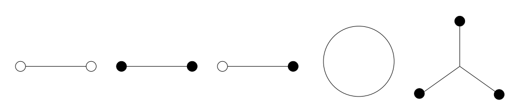

Figure 7: From left to right: the open interval, the closed interval, the half-open interval, the circle, and a bifurcation.

open unit disk, which is the set of points in $\mathbb{R}^2$ at distance less than one from the origin. This disk can be stretched over the entire plane. Define $f(x) = \frac{x}{1-\|x\|}$, which maps x to the point on the same radiating half-line at distance $\frac{1}{1-\|x\|}$ from the origin. Function f is bijective, continuous, and its inverse is continuous. It follows that the open disk is homeomorphic to $\mathbb{R}^2$. More generally, every open k-dimensional ball is homeomorphic to $\mathbb{R}^k$.

Triangulations. Unfortunately, the term 'triangulation' means something different in topology than in geometry. The topological notion of a triangulation is similar to the idea of a mesh in the sense that it imposes a combinatorial structure on a continuous domain.

Let K be a simplicial complex in $\mathbb{R}^d$. Its underlying space is a topological subspace of $\mathbb{R}^d$. K is a *triangula-*

tion of a topological space $\mathbb{X}$ if its underlying space is homeomorphic to the space: $|K| \approx \mathbb{X}$. $\mathbb{X}$ is *triangulable* if it has a triangulation. According to the definition, only closed spaces have triangulations. An example is the triangulation of the closed disk $\mathbb{B}^2$ with nine triangles, see Figure 8.

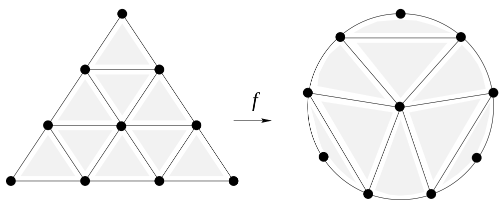

Figure 8: Triangulation of the closed disk. The homeomorphism maps each vertex, edge, triangle to a homeomorphic subset of the disk.

Manifolds. Manifolds are particularly simple topological spaces. They are defined locally. A *neighborhood* of a point $x \in \mathbb{X}$ is an open set that contains x. There are many neighborhoods and usually it suffices to take one that is sufficiently small. A topological space $\mathbb{X}$ is a *k-manifold* if every $x \in \mathbb{X}$ has a neighborhood homeomorphic to $\mathbb{R}^k$. It is probably more intuitive to mentally substitute a small open k-ball for $\mathbb{R}^k$, but this makes no difference because the two are homeomorphic.

A simple example of a k-manifold is the *k-sphere*, which is the set of points at unit distance from the origin in the $(k+1)$-dimensional Euclidean space:

$$\mathbb{S}^k \quad = \quad \{x \in \mathbb{R}^{k+1} \mid \|x\| = 1\},$$

see Figure 9. The smallest triangulation of $\mathbb{S}^k$ is the boundary complex of a $(k+1)$-simplex σ. To construct a homeomorphism we place σ so it contains the origin in its interior, and we centrally project every point of σ's boundary to the sphere.

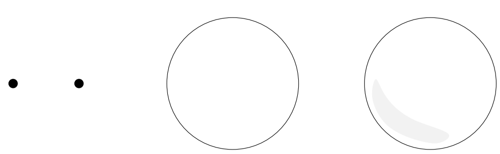

Figure 9: The 0-sphere is a pair of points, the 1-sphere is a circle, and the 2-sphere is what we usually call a sphere.

Manifolds with boundary. All points of a manifold have the same neighborhood. We get a more general class of spaces if we allow two types of neighborhoods. The second type of neighborhood is half an open ball. Again we can stretch the open half-ball, this time over half the Euclidean space of the same dimension. Formally, the *k-dimensional half-space* is

$$\mathbb{H}^k \quad = \quad \{x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k \mid x_1 \geq 0\}.$$

A space $\mathbb{X}$ is a *k-manifold with boundary* if every point $x \in \mathbb{X}$ has a neighborhood homeomorphic to $\mathbb{R}^k$ or to $\mathbb{H}^k$. The *boundary* is the set of points with neighborhood homeomorphic to $\mathbb{H}^k$, and it is denoted as $\text{bd}\,\mathbb{X}$. The boundary is always either empty or a $(k-1)$-manifold (without boundary). Why is that true? Note the slight awkwardness of language: a manifold with boundary is in general not a manifold, but a manifold is always a manifold with boundary, namely with empty boundary. An example of a k-manifold with (non-empty) boundary is the k-ball, see Figure 10. Its boundary is the $(k-1)$-sphere, $\text{bd}\,\mathbb{B}^k = \mathbb{S}^{k-1}$.

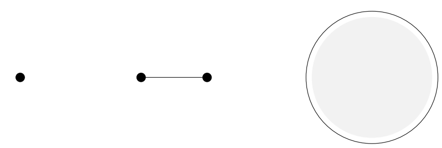

Figure 10: The 0-ball is a point, the 1-ball is a closed interval, and the 2-ball is a closed disk.

Orientability. Manifolds with or without boundary can be either orientable or non-orientable. The distinction is a global property that cannot be observed locally. Intuitively, we can imagine a $(k+1)$-dimensional ant walking on the k-manifold. At any moment the ant is on one side of the local neighborhood with which it is in contact. The manifold is *non-orientable* if there is a walk that brings the ant back to the same neighborhood but now on the other side, and it is *orientable* if no such path exists. Examples of non-orientable manifolds, one with and one without boundary, are the Möbius strip and the Klein bottle illustrated in Figure 11.

Imagine the boundary of a solid shape in our everyday three-dimensional space. This boundary is a 2-manifold and it bounds the interior of the shape on one side and the exterior on the other. The 2-manifold must therefore be orientable. At it turns out, every 2-manifold embedded in $\mathbb{R}^3$ separates inside from outside and is therefore orientable. The Klein bottle is non-orientably and cannot be embedded in $\mathbb{R}^3$. Any attempt to embed it produces self-intersections, such as the handle that passes through the side of the mug in Figure 11. On

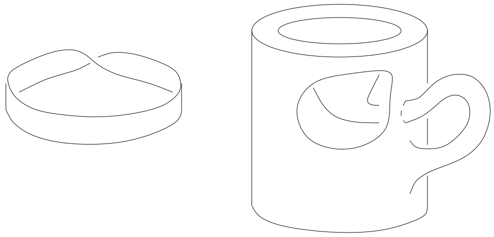

Figure 11: The Möbius strip to the left is bounded by a single circle. The Klein mug to the right is drawn with cut-away view to show a piece of the handle after it passes through the surface of the mug.

the other hand, there are obviously 2-manifolds with boundary that can be embedded in $\mathbb{R}^3$, and the Möbius strip is one example.

Bibliographic notes. Point set topology or general topology is an old and well-established branch of Mathematics. A good introductory text is the book by James Munkres. Manifolds are studied primarily in the context of differential structures. The topological aspects of such structures are emphasized in the text by Guillemin and Pollack [1]. The difference between orientable and non-orientable manifolds is discussed in a delightful book by Jeffrey Weeks [3].

[1] V. Guillemin and A. Pollack. *Differential Topology.* Prentice Hall, Englewood Cliffs, New Jersey, 1974.

[2] J. R. Munkres. *Topology. A First Course.* Prentice Hall, Englewood Cliffs, New Jersey, 1975.

[3] J. R. Weeks. *The Shape of Space.* Marcel Dekker, New York, 1985.