Simplicial Complexes

Topics: simplices, simplicial complexes, abstract simplicial complexes, geometric realizations, nerves.

Simplices. Points, edges, triangles, and tetrahedra are low-dimensional examples of simplices. We use combinations of points to define simplices in general dimensions. Let S be a finite set in d-dimensional Euclidean space denoted as $\mathbb{R}^d$. An *affine combination* of the points $p_i \in S$ is a point $x = \sum \alpha_i p_i$ with $\sum \alpha_i = 1$. The *affine hull*, $\text{aff}\, S$, is the set of all affine combinations. Equivalently, it is the intersection of all hyperplanes that contain S. The points in S are *affinely independent*, of *a. i.*, if none is the affine combination of the other points in S. For example, the affine hull of three a. i. points is a plane, that of two a. i. points is a line, and the affine hull of a single point is the point itself.

A *convex combination* is an affine combination with non-negative coefficients: $\alpha_i \geq 0$ for all $p_i \in S$. The *convex hull*, $\text{conv}\, S$, is the set of all convex combinations. Equivalently, it is the intersection of all half-spaces that contain S. A *simplex* is the convex hull of a set of a. i. points. If $S \subseteq \mathbb{R}^d$ is a set of $k+1$ a. i. points then the *dimension* of the simplex $\sigma = \text{conv}\, S$ is $\dim \sigma = k$ and σ is called a *k-simplex*. The largest number of a. i. points in $\mathbb{R}^d$ is $d+1$, and we have simplices of dimension -1 through d. Figure 1 shows the five types of simplices in $\mathbb{R}^3$. The points in any subset $T \subseteq S$ are a. i., so the convex hull of T is again a simplex. Specifically, $\tau = \text{conv}\, T$ is the subset of points $x \in \sigma$ with $\alpha_i = 0$ whenever $p_i \in S$ is not in T. The simplex τ is a *face* of σ, and we denote this relation by $\tau \leq \sigma$. If $\dim \tau = \ell$ then τ is called an *ℓ-face*. $\tau = \emptyset$ and $\tau = \sigma$ are *improper* faces and all others are *proper* faces of σ. The number of ℓ-faces of σ is equal to the number of ways we can choose $\ell+1$ from $k+1$ points, which is

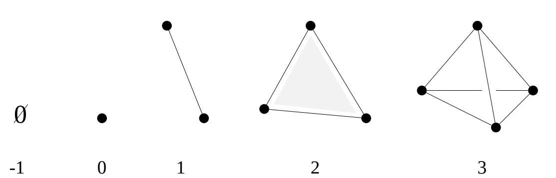

Figure 1: The (-1)-simplex is the empty set, a 0-simplex is a point or vertex, a 1-simplex is an edge, a 2-simplex is a triangle, and a 3-simplex is a tetrahderon.

$$\binom{k+1}{\ell+1} = \frac{(k+1)!}{(\ell+1)!(k-\ell)!}.$$

The total number of faces is

$$\sum_{\ell=-1}^{k} \binom{k+1}{\ell+1} = 2^{k+1}.$$

Simplicial complexes. A *simplicial complex* is a finite collection K of simplices such that

(i) $\sigma \in K$ and $\tau \leq \sigma$ implies $\tau \in K$, and

(ii) $\sigma, \upsilon \in K$ implies $\sigma \cap \upsilon \leq \sigma, \upsilon$.

Note that $\emptyset$ is a face of every simplex and thus belongs to K by condition (i). Condition (ii) therefore allows for the possibility that σ and υ be disjoint. Figure 2 shows three sets of simplices that violate one of the two conditions and therefore do not form complexes. A *subcomplex* is a subset that is a simplicial complex itself. Observe that every subset of a simplicial complex satisfies condition (ii). To enforce condition (i) we add

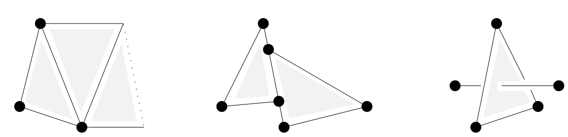

Figure 2: To the left we are missing an edge and two vertices. In the middle the triangles meet along a segment that is not an edge of either triangle. To the right the edge crosses the triangle at an interior point.

faces and simplices to the subset. Formally, the *closure* of a subset $L \subseteq K$ is the smallest subcomplex that contains L:

$$\mathrm{Cl}\, L = \{\tau \in K \mid \tau \leq \sigma \in L\}.$$

A particular subcomplex is the *i-skeleton* that consists of all simplices $\sigma \in K$ whose dimension is i or less. The *vertex set* is

$$\mathrm{Vert}\, K = \{\sigma \in K \mid \dim \sigma = 0\};$$

it is the same as the 0-skeleton except it does not contain the (-1)-simplex. The *dimension* of K is the largest dimension of any simplex: $\dim K = \max\{\dim \sigma \mid \sigma \in K\}$. If $k = \dim K$ then K is a *k-complex* and it is *pure* if every simplex is a face of a k-simplex. The *underlying space* is the set of points covered by simplices: $|K| = \bigcup K = \bigcup_{\sigma \in K} \sigma$. A *polyhedron* is the underlying space of a simplicial complex. Sometimes it is useful

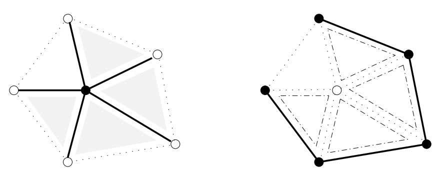

Figure 3: Star and link of a vertex. To the left the solid edges, and shaded triangles belong to the star of the solid vertex. To the right the solid edges and vertices belong to the link of the hollow vertex.

to consider substructures of a simplicial complex. The *star* of a simplex τ consists of all simplices that contain τ, and the *link* consists of all faces of simplices in the star that do not intersect τ:

$$\mathrm{St}\, \tau = \{\sigma \in K \mid \tau \leq \sigma\},$$
$$\mathrm{Lk}\, \tau = \{\sigma \in \mathrm{Cl}\, \mathrm{St}\, \tau \mid \sigma \cap \tau = \emptyset\},$$

see Figure 3 for examples. The star is generally not closed, but the link is always a simplicial complex.

Abstract simplicial complex. If we replace each simplex in a simplicial complex by the set of its vertices, we get a system of subsets of the vertex set. In doing so we throw away the geometry of the simplices, which allows us to focus on the combinatorial structure. Formally, a finite system A of finite sets is an *abstract simplicial complex* if $\alpha \in A$ and $\beta \subseteq \alpha$ implies $\beta \in A$. This requirement is similar to condition (i) for geometric simplicial complexes. A set $\alpha \in A$ is an *(abstract) simplex* and its dimension is $\dim \alpha = \mathrm{card}\, \alpha - 1$. The *vertex set* of A is $\mathrm{Vert}\, A = \bigcup A = \bigcup_{\alpha \in A} \alpha$. The concepts of face, subcomplex, closure, star, link extend straightforwardly from geometric to abstract simplicial complexes.

The set system together with the inclusion relation forms a partially ordered set, or poset, denoted as $(A, \subseteq)$. Posets are commonly drawn using Hasse diagrams where sets are nodes, smaller sets are below larger sets, and inclusions are edges, see Figure 4. Inclusions implied by others are usually not drawn. To

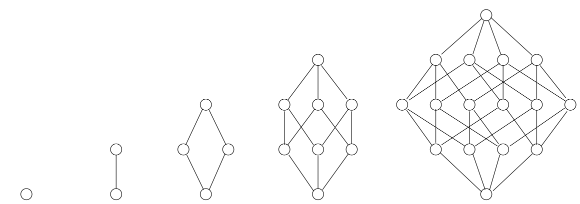

Figure 4: From left to right the poset of the empty set, a vertex, an edge, a triangle, a tetrahedron.

draw the Hasse diagram of a k-simplex α we draw the Hasse diagrams for two $(k-1)$-simplices. One is the diagram of a $(k-1)$-face β of α and the other is the diagram for the star of the vertex $u \in \alpha - \beta$. Finally, we connect every simplex γ in the star of u with the simplex $\gamma - \{u\}$ in the closure of β. An abstract simplicial complex A is a subsystem of the power set of $\mathrm{Vert}\, A$. We can therefore think of it as a subcomplex of an n-simplex, where $n + 1 = \mathrm{card}\, \mathrm{Vert}\, A$. This view is expressed in the picture of an abstract simplicial complex shown as Figure 5.

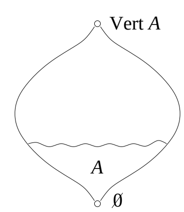

Figure 5: The onion is the power set of $\mathrm{Vert}\, A$. The area below the waterline is an abstract simplicial complex.

Geometric realization. We can think of an abstract simplicial complex as an abstract version of a geometric simplicial complex. To formalize this idea we define a *geometric realization* of an abstract simplicial complex A as a simplicial complex K together with a bijection $\varphi : \mathrm{Vert}\, A \to \mathrm{Vert}\, K$ such that $\alpha \in A$ iff $\mathrm{conv}\, \varphi(\alpha) \in K$. Conversely, A is called an *abstraction* of K.

Given A, we can ask for the lowest dimension that allows a geometric realization. For example, graphs are 1-dimensional abstract simplicial complexes and can always be realized in $\mathbb{R}^3$. Two dimensions are sometimes but not always sufficient. This result generalized to k-dimensional abstract simplicial complexes: they can always be realized in $\mathbb{R}^{2k+1}$ and sometimes $\mathbb{R}^{2k}$ does not suffice. To prove the sufficiency of the claim we show that the k-skeleton of every n-simplex can be realized in $\mathbb{R}^{2k+1}$. Map the $n+1$ vertices to points in general position in $\mathbb{R}^{2k+1}$. Specifically, we require that any $2k+2$ of the points are a. i. Two simplices σ and υ of the k-skeleton have a total of at most $2(k+1)$ vertices, which are therefore a. i. In other words, σ and υ are faces of a common simplex of dimension at most $2k+1$. It follows that $\sigma \cap \upsilon$ is a common face of both.

THEOREM 1. Every k-dimensional abstract simplicial complex has a geometric realization in $\mathbb{R}^{2k+1}$.

Nerves. A convenient way to construct abstract simplicial complexes uses arbitrary finite sets. The *nerve* of such a set C is the system of subsets with non-empty intersection:

$$\mathrm{Nrv}\, C = \{X \subseteq C \mid \bigcap X \neq \emptyset\}.$$

If $Y \subseteq X$ then $\bigcap X \subseteq \bigcap Y$. Hence $X \in \mathrm{Nrv}\, C$ implies $Y \in \mathrm{Nrv}\, C$, which shows that the nerve is an abstract simplicial complex. Consider for example the case where C is a covering of some geometric space, such as in Figure 6. Every set in the covering corresponds

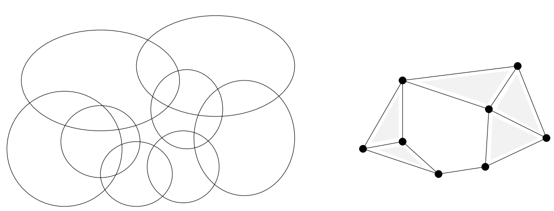

Figure 6: A covering with eight sets to the left and its nerve to the right. The sets meet in triplets but not in quadruplets, which implies that the nerve is 2-dimensional.

to a vertex, and $k+1$ sets with non-empty intersection define a k-simplex.

We have seen an example of such a construction earlier. The Voronoi regions of a finite set $S \subseteq \mathbb{R}^2$ define a covering $C = \{V_a \mid a \in S\}$ of the plane. Assuming general position the Voronoi regions meet in pairs and in triplets, but not in quadruplets. The nerve therefore consists only of abstract vertices, edges, and triangles. Consider the function $\varphi : C \to \mathbb{R}^2$ that maps a Voronoi region to its generator: $\varphi(V_a) = a$. This function defines a geometric realization of $\mathrm{Nrv}\, C$:

$$D = \{\mathrm{conv}\, \varphi(\alpha) \mid \alpha \in \mathrm{Nrv}\, C\}.$$

This is the Delaunay triangulation of S. What happens if the points in S are not in general position? If $k+1 \geq 4$ Voronoi regions have a non-empty common intersection then $\mathrm{Nrv}\, C$ contains the corresponding abstract k-simplex. So instead of making a choice among the possible triangulations of the $(k+1)$-gon, the nerve takes all possible triangulations together and interprets them as subcomplexes of a k-simplex. The disadvantage of this method is of course the fact that $\mathrm{Nrv}\, C$ can no longer be realized in $\mathbb{R}^2$.

Bibliographic notes. During the first half of the twentieth century, combinatorial topology was a flourishing field of Mathematics. We refer to Paul Alexandrov [1] as a comprehensive text originally published as a series of three books. This text roughly coincides with a fundamental reorganization of the field triggered by a variety of technical results in topology. One of the successors of combinatorial topology is modern algebraic topology where the emphasis shifts from combinatorial to algebraic structures. We refer to James Munkres [5] for an introductory text in that area.

We proved that every k-complex can be geometrically realized in $\mathbb{R}^{2k+1}$. Examples of k-complexes that require $2k+1$ dimensions are provided by Flores [2] and independently by van Kampen [3]. One such example is the k-skeleton of the $(2k+2)$-simplex. For $k=1$ this is the complete graph of five vertices, which is one of the two obstructions of graph planarity identified by Kuratowski [4].

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