Homework Assignment

The solution for each question should fit on one side of a single piece of paper.

1. **Nested subgraphs.** Let S be a finite set of points in general position in $\mathbb{R}^2$. Consider the minimum spanning tree, the relative neighborhood graph, the Gabriel graph, and the Delaunay triangulation of S. Show that

$$MST \subseteq RNG \subseteq GG \subseteq D.$$

   Is it possible that one of the graphs is not connected?

2. **Links in alpha complexes.** Let $S \subseteq \mathbb{R}^2$ be a finite set of points in general position. For a fixed $\alpha > 0$ let K be the α-complex and U the union of open α-disks centered at points of S.

   (i) The link of an edge $pq \in K$ contains 0, 1, or 2 vertices. How are the three cases reflected in the boundary of U?

   (ii) The link of a vertex $p \in K$ is either a circle, an interval, a point, or a union of intervals and points. How are the cases reflected in the boundary of U?

3. **Boundary of disk union.** Let U be the union of n closed unit disks in general position in $\mathbb{R}^2$. The boundary $\mathrm{bd}\, U$ of U consists of one or more components. Each component is a cyclic list of circular arcs.

   (i) Show that $\mathrm{bd}\, U$ has fewer than $2n$ components.

   (ii) Show that each component consists of fewer than $2n$ arcs.

   (iii) Show that the total number of arcs in $\mathrm{bd}\, U$ is less than $6n$.

4. **Another subcomplex.** Let D be the Delaunay triangulation of a finite set S of points in general position in $\mathbb{R}^2$. For a vertex, edge, triangle $\sigma \in D$ let $\varrho(\sigma)$ be the radius of the smallest closed disk that contains σ. Consider the set of simplices covered by disks of radius at most $\alpha > 0$:

$$L(\alpha) = \{\sigma \in D \mid \varrho(\sigma) \leq \alpha\}.$$

   Show that $L(\alpha)$ is a simplicial complex, and that the α-complex is a subcomplex, $K(\alpha) \subseteq L(\alpha)$.

5. **Crust definitions.** Let S be a finite set of points in $\mathbb{R}^2$ and V the set of Voronoi vertices. Assume no four points of $S \cup V$ are cocircular. Prove that under this assumption the following definitions of the *crust* of S are equivalent:

   (i) the collection of simplices in the Delaunay triangulation of $S \cup V$ whose vertices all belong to S,

   (ii) the intersection of the Delaunay triangulation of $S \cup V$ and the Delaunay triangulation of S,

   (iii) the Delaunay triangulation of $S \cup V$ restricted by the union of Voronoi regions generated by points in S.