Homework Assignment

The solution for each question should fit on one side of a single piece of paper.

1. **Stars and links.** Let K be a 2-complex that triangulates the closed disk, $\mathbb{B}^2$. Let a and b be interior vertices, u, v, w boundary vertices, ab, au, uv interior edges, and vw a boundary edge. Draw K such that it contains (among others) five vertices and four edges as specified. Furthermore draw the star and the link of each of the following subsets of K: $\{a\}$, $\{u\}$, $\{ab\}$, $\{ab, a, b\}$, $\{au, a, u\}$, $\{uv, u, v\}$, $\{vw\}$, $\{vw, v, w\}$.

2. **Irreducibility.** A 2-complex K is *irreducible* if the contraction of ab changes the topological type of K for every edge $ab \in K$. Prove that the only irreducible triangulation of the 2-sphere is the boundary complex of the tetrahedron.

3. **Vertex map.** Construct a bijective vertex map $g : \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ such that the simplicial map $\psi : |K| \to |L|$ that extends g is not a homeomorphism.

4. **Necessity of link condition.** Let K be the triangulation of a 2-manifold. Recall Theorem 2A, which says that the contraction of an edge $ab \in K$ preserves the topological type iff $\mathrm{Lk}\, a \cap \mathrm{Lk}\, b = \mathrm{Lk}\, ab$. We proved the sufficiency of the link condition in the lecture on simplicial maps. Prove the necessity of the link condition. In other words, prove that $\mathrm{Lk}\, a \cap \mathrm{Lk}\, b \neq \mathrm{Lk}\, ab$ implies that the contraction of ab changes the topological type of the 2-manifold.

5. **Square distance minimization.** Let S be a finite set of points in $\mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ map each point x in the plane to the sum of square distances:

$$f(x) \quad = \quad \sum_{p \in S} \|x - p\|^2.$$

   (i) Show that f is a quadratic function and that it has a unique minimum.

   (ii) Given two finite sets $S_1, S_2 \subseteq \mathbb{R}^2$ together with their maps f_1, f_2, show how to compute the map f for $S = S_1 \cup S_2$ in constant time.

   (iii) How would you generalize your answer to (ii) if you are given a constant number of finite sets S_i and maps f_i?

6. **Points, lines, and planes.** Let S be a finite set of points in $\mathbb{R}^2$.

   (i) Construct a finite set H of lines such that the sum of square distances is the same for every point x in the plane:

$$\sum_{p \in S} \|x - p\|^2 \quad = \quad \sum_{\ell \in H} \|x - \ell\|^2,$$

   where $\|x - \ell\| = \min\{\|x - y\| \mid y \in \ell\}$.

   (ii) Are the lines in your answer to (i) unique?

   (iii) Generalize your solution to the case where S is a finite set of lines and H is a finite set of planes in $\mathbb{R}^3$.