Homework Assignment

The solution for each question should fit on one side of a single piece of paper.

1. **Union of intervals.** Let I be a set of n closed intervals on the real line. The union of the intervals is a collection of $m \leq n$ pairwise disjoint intervals. Prove that

$$m = \sum_{i \geq 0} (-1)^i \cdot n_i,$$

where n_i counts the $(i+1)$-tuplets of intervals with non-empty common intersection.

(We can state the same question in different words. Let m be the number of components of $\bigcup I$. Then

$$m = \sum_{J \in \mathrm{Nrv}\, I} (-1)^{\dim J},$$

where $\mathrm{Nrv}\, I = \{J \subseteq I \mid \bigcap J \neq \emptyset\}$.)

2. **Union of simplices.** The union of a finite collection of triangles in $\mathbb{R}^2$ is a polyhedron because we can find another set of triangles with the same union whose closure is a simplicial complex. Show that the natural extension of this statement to $\mathbb{R}^d$ is true.

3. **Extendable shelling.** Let K be a triangulation of the closed disk, let n be the number of triangles, and let $\tau_1, \tau_2, \ldots, \tau_i$ be an ordering of $i < n$ triangles so every prefix defines another triangulation of $\mathbb{B}^2$. For obvious reasons the sequence from τ_1 through τ_i is called a *partial shelling* of K. Prove that every partial shelling can be extended to a complete shelling

$$\tau_1, \tau_2, \ldots, \tau_i, \tau_{i+1}, \ldots, \tau_n.$$

4. **Shelling algorithm.** Let K be a 2-complex that triangulates $\mathbb{B}^2$ with n triangles. Describe an algorithm that constructs a shelling of K in time $O(n)$. You may assume that K is given in a data structure that is convenient for your algorithm.

5. **Classifying manifolds.** Let K be a triangulation of a 2-manifold.

   (i) Give a linear time algorithm that decides whether or not the 2-manifold is orientable.

   (ii) Assuming it is orientable, give a linear time algorithm that computes the genus of the 2-manifold.

6. **2-tree.** A tree can be defined as a pure 1-complex whose edges can be ordered such that each but the first edge shares exactly one vertex with its predecessors. A *2-tree* is a pure 2-complex whose triangles can be ordered such that each but the first triangle shares exactly one edge and the two endpoints of that edge with its predecessors.

   (i) Show that a 2-tree with n triangles has $n+2$ vertices and $2n+1$ edges.

   (ii) Show that the link of every vertex is a tree.

   (iii) Define d-trees, for every $d \geq 1$.

7. **Euler characteristic.** Compute the Euler characteristic of a the following topological spaces.

   (i) An annulus in the plane.

   (ii) A Möbius strip.

   (iii) The Klein bottle.

   (iv) A solid torus.

   (v) The boundary complex of each Platonic solid (tetrahedron, cube, octahedron, dodecahedron, icosahedron).

8. **Embedding the dunce cap.** Recall that the dunce cap can be construction from a triangular sheet of paper by gluing all three edges to one. Does the dunce cap have a triangulation that can be geometrically realized in $\mathbb{R}^3$?