Homework Assignment

The solution for each question should fit on one side of a single piece of paper.

1. **Section of triangulation.** Let K be a triangulation of a set of n points in the plane. Let ℓ be a line that avoids all point. Prove that ℓ intersects at most $2n-4$ edges of K and that this upper bound is tight for every $n \geq 3$.

2. **Minimum spanning tree.** The notion of minimum spanning tree can be extended from weighted graphs to a geometric setting where the nodes are points in the plane. Take the complete graph of the set of nodes and define the length of an edge as the Euclidean distance between its endpoints. A minimum spanning tree of that graph is a *Euclidean minimum spanning tree* of the point set. Prove that all edges of every Euclidean minimum spanning tree belong to the Delaunay triangulation of the same point set.

3. **Acute triangles.** An *acute* triangle has all three angles less than $\frac{\pi}{2}$. Note that a Delaunay triangle abc is acute iff the dual Voronoi vertex is contained in the interior of abc. Show that a triangulation K all of whose triangles are acute is necessarily the Delaunay triangulation of its vertex set.

4. **Gabriel graph.** Let S be a finite set in the plane. Let $a, b \in S$ and consider the smallest circle C_{ab} that passes through a and b. The edge ab belongs to the *Gabriel graph* G of S if C_{ab} is empty and a, b are the only two points of S that lie on the circle.

   (i) Show that $ab \in G$ iff ab belongs to the Delaunay triangulation and the opposite angles in the one or two triangles that share ab are less than $\frac{\pi}{2}$.

   (ii) Show that $ab \in G$ iff ab crosses the dual Voronoi edge $V_a \cap V_b$.

5. **Maximum sorted angle vector.** Let K be a triangulation of a finite set in the plane. Let t be the number of triangles and consider the sorted vector of angles,

$$v(K) \quad = \quad (\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{3t}).$$

   Let L be another triangulation of the same point set and call L *smaller than or equal to* K if $v(L) = v(K)$ or $v(L)$ is lexicographically smaller than $v(K)$. Prove that all triangulations are smaller than or equal to the Delaunay triangulation.

6. **Minmax circumcircle.** Let K be a triangulation of a finite set in the plane and let $\varrho(K)$ be the radius of the largest circumcircle of any triangle in K. Prove $\varrho(K) \geq \varrho(D)$, where D is the Delaunay triangulation of the set.

7. **Random permutation.** Show that the following algorithm constructs a random permutation of the integers 1 through n.

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for i = 1 to n do
  Z[i] = i; choose random index 1 ≤ j ≤ i;
  swap Z[i] and Z[j]
endfor.
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8. **Line segments.** Consider a set L of n pairwise disjoint line segments in the plane. The *distance* of a point $x \in \mathbb{R}^2$ from $ab \in L$ is defined as the minimum Euclidean distance from any point on ab:

$$\|x - ab\| \quad = \quad \min\{\|x - y\| \mid y \in ab\}.$$

   The *Voronoi region* of ab is then the set of points x for which ab is no further than any other line segment in L.

   (i) Prove that the Voronoi region of every line segment in L is connected.

   (ii) Show that the edges of the Voronoi regions are pieces of lines and parabolas.

   (iii) Prove that that the number of Voronoi edges is at most some constant times n.