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                        CS 61B  Lab 9
                      March 19-20, 2009
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Goal: to practice proving asymptotic (big-Oh) results. The class notes from
Lecture 20 may come in handy; however, we will try to be even more rigorous
here than in lecture.

Recall the definition of big-Oh:

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|==============================================================================|
|  O(f(n)) is the SET of ALL functions T(n) that satisfy:                       |
|                                                                               |
|     There exist positive constants c and N such that, for all n >= N,         |
|                            T(n) <= c f(n)                                      |
|==============================================================================|
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EXAMPLE

Formally prove that $n^2 + n + 1$ is in $O(n^2)$.

Solution:
  Let $T(n) = n^2 + n + 1$. Let $f(n) = n^2$.

  Choose $c = 3$, and $N = 1$. Then, we know $T(n)$ is in $O(n^2)$ if we can prove

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                        T(n) <= c f(n),
   or equivalently,    n^2 + n + 1 <= 3 n^2,                 for all n >= 1.
```

  Is this inequality true? Well, for any $n >= 1$, we know that $1 <= n <= n^2$.
  Hence, all of the following are true:

$$1 <= n^2$$
$$n <= n^2$$
$$n^2 = n^2$$

  Adding the left and right sides of these inequalities together, we have

$$n^2 + n + 1 <= 3 n^2,\ \text{which completes the proof.}$$

Part I: (1 point)

Formally prove that $2^n + 1$ is in $O(4^n - 16)$.

HINT: You may assert without proof that, for all $n >= 1$, $2^n >= 1$.
  (You may also assert without proof that 4^n and 2^n are monotonically
  increasing, if you find it useful.)

Part II: (1 point)

Formally prove that if $f(n)$ is in $O(g(n))$, and $g(n)$ is in $O(h(n))$, then
  $f(n)$ is in $O(h(n))$.

NOTE: The values of c and N used to prove that $f(n)$ is in $O(g(n))$ are
  not necessarily the same as the values used to prove that $g(n)$ is in $O(h(n))$.
  Hence, assume that there are positive c', N', c'', and N'' such that

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   f(n) <= c' g(n)         for all n >= N', and
   g(n) <= c'' h(n)        for all n >= N''.
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Part III: (2 points)

Formally prove that $0.01 n^2 - 1$ is NOT in $O(n)$.

We need to show that, no matter how large we choose c and N, we will never
  obtain the desired inequality. We cannot prove this by picking a specific
  value of c and N. Instead, we must study how the two functions behave as
  n approaches infinity.

Let $T(n) = 0.01 n^2 - 1$, and let $f(n) = n$. Prove that

$$\lim_{n \to \infty} \frac{c\, f(n)}{T(n)} = 0,$$

  no matter how large we choose c to be. You will need to scale both the
  numerator and the denominator by a well-chosen multiplier to get the result.

Use this result to show that there are no values c, N such that $T(n) <= c f(n)$
  for all $n >= N$.

Postscript

The functions $|\cos(n)|$ and $|\sin(n)|$ are interesting, because neither is
  dominated by the other. Can you informally suggest why $|\cos(n)|$ is not in
  $O(|\sin(n)|)$, and $|\sin(n)|$ is not in $O(|\cos(n)|)$?