CS 274
Computational Geometry (Autumn 2009)
Homework 5

Homework 5 is due **at the start of class (2:40 pm) on Wednesday, December 2, 2009**.

You may use algorithms learned in class as subroutines without re-explaining them.

[1] General-dimensional Voronoi diagrams (4 points). The worst-case complexity of a Voronoi diagram is $\Theta(n^2)$ in three dimensions and $\Theta(n^{\lceil d/2 \rceil})$ in E^d.

(i) Prove that a planar cross-section of a d-dimensional Voronoi diagram—in other words, the planar subdivision formed by intersecting a 2-flat with a Voronoi diagram—has complexity no greater than $\mathcal{O}(n)$. Hint: what do you know about the properties of Voronoi cells?

(ii) Does a planar cross-section of a d-dimensional Delaunay triangulation also always have a complexity in $\mathcal{O}(n)$? Explain.

[2] Deterministic BSPs (4 points). Consider this deterministic algorithm for constructing a BSP tree for a set of n non-crossing line segments in the plane. Suppose you are deciding how to slice a cell C of the BSP subdivision. (When we start, C is the entire plane.) If there is a free split, take it. Otherwise, make a list of the segment endpoints (not fragment endpoints, just endpoints of the original segments) in C, and draw a vertical splitting line through the endpoint with median x-coordinate (breaking ties arbitrarily).

Prove that this algorithm always constructs a BSP tree with at most $\mathcal{O}(n \log n)$ fragments.

[3G] Minkowski sums (7 points). Suppose we want to automatically machine a two-dimensional part by cutting away the region around it with a circular drill bit. Let D be a disk whose center is the origin and whose radius r is the radius of the drill bit. When machining a shape S, we may place the center of the drill bit at any point not in the interior of $S \oplus D$.

We wish to argue that for any n-vertex simple polygon P (not necessarily convex), the worst-case complexity of $P \oplus D$ is in $\mathcal{O}(n)$. The complexity is the total number of vertices, line segments, and circular arcs needed to represent $P \oplus D$. Line segments are induced by edges of P, offset sideways by the radius r of D; and circular arcs (of radius r) are induced by vertices of P.

Because $P \oplus D$ is a planar graph, it suffices to bound the number of vertices. There are three types of vertices that can appear: intersections between two line segments, intersections between two arcs, and intersections between a line segment and an arc.

(i) Demonstrate that a single edge of P can induce up to $\Theta(n)$ line segments in $P \oplus D$, and that a single vertex of P can induce up to $\Theta(n)$ circular arcs. (A couple of simple figures should suffice to illustrate these two points—no words are necessary.)

(ii) An edge e of P can induce two sequences of line segments in $P \oplus D$, offset r to either side of e. Suppose two line segments of $P \oplus D$, s_1 and s_2, intersect at a vertex v. Argue that v terminates one of the sequences of line segments induced by some edge of P. Use this fact to argue that $P \oplus D$ can have only $\mathcal{O}(n)$ vertices formed by pairs of line segments.

(iii) Use the edges of a Voronoi diagram to argue that $P \oplus D$ can have only $\mathcal{O}(n)$ vertices formed by pairs of arcs.

Note: the number of segment-arc intersections can be bounded by the same reasoning as part (iii), by using the properties of the Voronoi diagram of the line segments. (See Section 7.3 of the third edition of the Dutch Book if you're curious what that is.)

[4] Overlapping intervals (7 points).

(i) Let S be a set of n intervals. Describe a data structure that uses $\mathcal{O}(n)$ storage and returns the *number* of intervals in S that intersect any query interval I in $\mathcal{O}(\log n)$ time. (Note that I didn't say "$\mathcal{O}(k+\log n)$ time.") Hint: I can think of two very different ways to do this, one of which modifies one of the data structures taught in class, but I can't think of a way that uses an interval tree.

(ii) Explain how to use a two-dimensional layered range tree to answer the same counting query, also in $\mathcal{O}(\log n)$ time. (Assume you have a correct solution to Problem [3] of Homework 4; there's no need to explain that solution again. Observe that a two-dimensional range tree generally requires $\Theta(n \log n)$ space, so this isn't an answer to part (i).)

(iii) Let D be a disk, and let L be a set of l lines in the plane. Describe an algorithm that returns, in $\mathcal{O}(l \log l)$ time, the number of pairs of lines that intersect each other inside D. For simplicity, you may assume that no two lines intersect each other precisely on the boundary of D. Hint: as best I can figure, this can be done with the range tree but not with the answer to part (i). But I'd love to be proven wrong.

[5G] Point-rectangle pairs (8 points). Let P be a set of m real numbers, and let R be a set of n intervals. By building an interval tree or a segment tree, we can report all pairs $\langle p \in P, r \in R \rangle$ such that $p \in r$ in $\mathcal{O}(k + (m+n)\log n)$ time (including preprocessing), where k is the number of such pairs.

Observe that k could be as large as $\Theta(mn)$. Suppose that instead of reporting pairs individually, we are allowed to report pairs of subsets of the form $\langle \{p_1, p_2, \ldots, p_i\}, \{r_1, r_2, \ldots, r_j\} \rangle$, meaning that each point $p_1 \ldots p_i$ lies in every interval $r_1 \ldots r_j$. These *subset pairs* sometimes make it possible to express the results in $o(k)$ space. Every intersecting real–interval pair $\langle p_i, r_j \rangle$ should appear in *exactly* one of the subset pairs in the output.

(i) Describe an algorithm that runs in $\mathcal{O}((m+n)\log n)$ time (including all preprocessing) and reports the real–interval pairs in $\mathcal{O}(\min\{k, (m+n)\log n\})$ space. (This is the amount of space taken by the subset pairs; the algorithm may use more.)

(ii) Improve your answer so that it runs in $\mathcal{O}(k' + m + n\log n)$ time, where k' is the size of the output. Hint: for each number in P, search the tree from bottom to top instead of top to bottom. To make this possible in the given time bound, you will need radix sort, and you will need to move items up the tree in batches rather than individually.

(iii) Let P be a set of m points in the plane, and let R be a set of n axis-parallel rectangles (possibly intersecting). We want to report all pairs $\langle p \in P, r \in R \rangle$ such that $p \in r$. Suppose there are k such pairs. Again we use subset pairs to save space and time. Describe an algorithm that reports the point–rectangle pairs in $k' \in \mathcal{O}(\min\{k, (m+n)\log^2 n\})$ space and runs in $\mathcal{O}(k' + m\log n + n\log^2 n)$ time.
You'll get **2 bonus points** if you can do it in $\mathcal{O}(k' + m + n\log^2 n)$ time. (I don't actually know if this is possible. It would take very careful organization of the batches of points to avoid getting charged $\log n$ for each point that is contained in few or no rectangles.)