CS 189/289A Introduction to Machine Learning
Spring 2024 Jonathan Shewchuk Final

- Please do not open the exam before you are instructed to do so. Fill out the blanks below now.
- **Electronic devices are forbidden on your person**, including phones, laptops, tablet computers, headphones, and calculators. Turn your cell phone off and **leave all electronics at the front of the room**, or **risk getting a zero** on the exam. Exceptions are made for car keys and devices needed because of disabilities.
- When you start, the **first thing you should do** is **check that you have all 12 pages and all 6 questions**. The second thing is to please **write your initials at the top right of every page after this one** (e.g., write "JS" if you are Jonathan Shewchuk).
- The exam is closed book, closed notes except your two cheat sheets.
- You have **180 minutes**. (If you are in the Disabled Students' Program program and have an allowance of 150% or 200% time, that comes to 270 minutes or 360 minutes, respectively.)
- Mark your answers on the exam itself in the space provided. Do **not** attach any extra sheets. If you run out of space for an answer, write a note that your answer is continued on page 5.
- The total number of points is 150. There are 18 multiple choice questions worth 4 points each, and 5 written questions worth a total of 78 points.
- For multiple answer questions, fill in the bubbles for **ALL correct choices.** There may be more than one correct choice, but there is always at least one correct choice. Partial credit rules: For questions with one correct bullet, 2/4 partial credit if you check the correct bullet plus exactly one other (wrong) bullet. For questions with two correct bullets: 2/4 partial credit if you have any 3 out of 4 bullets correct. For questions with three correct bullets: 2/4 partial credit if you check 2 out of 3 correct bullets but do not check the incorrect one. For questions with four correct bullets: 2/4 partial credit if you check 3 of the 4 bullets.

First name	
Last name	
SID	
Name and SID of student to your left	
Name and SID of student to your right	

Q1. [72 pts] Multiple Answer

Fill in the bubbles for **ALL correct choices.** There may be more than one correct choice, but there is always at least one correct choice. Partial credit rules: For questions with one correct bullet, 2/4 partial credit if you check the correct bullet plus exactly one other (wrong) bullet. For questions with two correct bullets: 2/4 partial credit if you have any 3 out of 4 bullets correct. For questions with three correct bullets: 2/4 partial credit if you check 2 out of 3 correct bullets but do not check the incorrect one. For questions with four correct bullets: 2/4 partial credit if you check 3 of the 4 bullets.

(a) [4 pts] **Why** do we usually choose splits that maximize the information gain—equivalently, minimize the weighted average of entropies $H_{\text{after}} = \frac{|S_l|H(S_l) + |S_r|H(S_r)}{|S_l| + |S_r|}$ after the split—**instead of** computing the number of misclassified points in each child and using the weighted average of those as a cost function?

○ A: The misclassified points cost function penalizes balanced splits (where the two children have roughly equal numbers of training points) more than the entropy cost function does.

○ B: The misclassified points cost function often assigns the same weighted average cost to several different splits.

○ C: The entropy cost is strictly concave, so if the children have different distributions of classes (and are nonempty), the information gain is positive.

○ D: The misclassified points cost is not strictly concave, so we encounter more circumstances where the weighted average of the children's costs is equal to the parent's cost.

(b) [4 pts] Select the true statements about **decision trees**.

○ A: Decision trees are grown from the bottom up, meaning that we start at the leaves and repeatedly fuse them until we reach the root node.

○ B: Pruning improves both training and validation accuracy.

○ C: They can guarantee 100% training accuracy on any training set where no two identical points have different labels.

○ D: Decision trees are used for classification only, not regression.

(c) [4 pts] Select the true statements about **random forests**.

○ A: Restricting the trees to be shallow decreases the bias.

○ B: A random forest decreases the variance (compared to a single decision tree).

○ C: We restrict which features can be used for splitting in each treenode to decrease the correlation between the trees' predictions.

○ D: In a typical tree in the forest, some training points are not used.

(d) [4 pts] Select the valid reasons why you might choose **not** to use **bagging** on a particular training set.

○ A: Your model has low bias and high variance.

○ B: Your model has high bias and low variance.

○ C: You want your model to be interpretable.

○ D: You are solving a regression problem.

(e) [4 pts] Select the true statements about *k***-nearest neighbor classification**.

○ A: As k increases, the decision boundary tends to look smoother.

○ B: In 3-class classification, choosing k to be prime ensures there will never be a tie between classes.

○ C: Even the slow, $\Theta(nd)$-time exhaustive nearest neighbor algorithm is often used in practice.

○ D: As the number of neighbors grows as $k \to \infty$ and the number of training points grows even faster, so $n/k \to \infty$, the k-nearest neighbor error rate converges to the Bayes risk.

(f) [4 pts] Select the valid reasons that **ReLUs may be preferred over sigmoids (logistic functions)** as activation functions for the **hidden** layers of a neural network.

○ A: The forward and backward passes are computationally cheaper with ReLUs than with sigmoids.

○ B: The cost function of a ReLU-based neural network, trained with the squared-error loss, will be convex because the ramp (ReLU) function is convex.

○ C: ReLUs are less vulnerable to the vanishing gradient problem than sigmoids.

○ D: The cost function of a ReLU-based neural network is smooth with a gradient defined everywhere in weight space, whereas the cost function of a sigmoid-based neural network is not.

(g) [4 pts] Which steps are customarily (usually) part of training a neural network's weights with **backpropagation**?

○ A: Computing the partial derivatives of each weight with respect to each weight in the previous layer.

○ B: Computing the partial derivatives of a cost function or loss function with respect to each weight.

○ C: Computing the partial derivatives of a cost function or loss function with respect to each hidden unit value.

○ D: Computing the partial derivatives of a cost function or loss function with respect to each input feature.

(h) [4 pts] Select the true statements about **principal components analysis** (PCA).

○ A: The first principal component is always orthogonal to the second principal component.

○ B: We use the subset of principal components associated with the smallest eigenvalues of the sample covariance matrix.

○ C: PCA requires the training points to have labels.

○ D: If the sample covariance matrix has an eigenvalue of zero, the corresponding eigenvector (principal component) is orthogonal to a hyperplane that passes through all the training points.

(i) [4 pts] Select the true statements about **the objective of principal components analysis** (PCA).

○ A: We want to find a subspace that minimizes the mean of the squared projection distances.

○ B: We want to find a subspace that maximizes the mean of the squared projection distances.

○ C: We want to find a subspace that minimizes the sample variance of the projected sample points.

○ D: We want to find a subspace that maximizes the sample variance of the projected sample points.

(j) [4 pts] Suppose that $\dot{X} \in \mathbb{R}^{n \times d}$ is a **centered** design matrix. Recall that its singular value decomposition (SVD) is written $\dot{X} = UDV^\top$. Select the true statements about **principal components analysis** (PCA) **and the SVD**.

○ A: The principal components are columns of V.

○ B: When k is much less than d, we can find k principal components faster by computing a partial SVD than we can by computing the eigenvectors of the sample covariance matrix.

○ C: The principal coordinates for sample point $\dot{X}_i$ appear in row i of V.

○ D: The diagonal entries of D are the eigenvalues that correspond to the principal components.

(k) [4 pts] While training your spam classifier in Homework 1, you observe that your soft-margin support vector machine's **training accuracy is only 58% and the validation accuracy is 54%**. Which interventions have a significant chance of making it possible to obtain **test accuracy** on Kaggle of 80% or higher?

○ A: Try smaller values of the hyperparameter C.

○ B: Delete features with poor predictive power.

○ C: Obtain more training data.

○ D: None of the above.

(l) [4 pts] Suppose we have two classes, A and B, and we wish to train a classifier to recognize them.

○ A: If we want to use decision theory (risk minimization) to create a generative model, the two classes must each have a normal distribution.

○ B: If the classes are not linearly separable and we want to use a soft-margin support vector machine, we must create added features or the training algorithm will fail to output a classifier.

○ C: If we have only one training point for class A and only one training point (at a different location) for class B, the centroid method classifier obtains 100% training accuracy.

○ D: If all the training points are at distinct locations, the 1-nearest neighbor algorithm obtains 100% training accuracy.

(m) [4 pts] Consider a least squares problem where we have a design matrix $X \in \mathbb{R}^{n\times d}$ (we don't use a bias term) and a vector of labels $y \in \mathbb{R}^n$. We wish to learn a weight vector $w \in \mathbb{R}^d$ that minimizes $\text{RSS}(w) = \|Xw - y\|^2$. If X is invertible, the least-squares solution is $\hat{w} = (X^\top X)^{-1}X^\top y$. Select the true statements.

○ A: $n \geq d$ is a necessary condition for $X^\top X$ to be invertible.

○ B: $n \geq d$ is a sufficient condition for $X^\top X$ to be invertible.

○ C: If X is invertible, then there is a solution w^* such that $\text{RSS}(w^*) = 0$.

○ D: If we instead use ridge regression with a positive λ, then ridge regression always has a unique solution regardless of the invertibility of $X^\top X$.

(n) [4 pts] Select the true statements about **AdaBoost**.

○ A: AdaBoost is an ensemble method designed expressly to reduce the variance of decision trees.

○ B: After enough iterations, AdaBoost can always obtain 100% training accuracy, regardless of what classifier it uses.

○ C: AdaBoost with decision trees typically uses different criteria/hyperparameters to build the trees than random forests do.

○ D: AdaBoost adjusts the weights of misclassified training points to increase their information gain.

(o) [4 pts] Select the true statements about **Lasso**.

○ A: The Lasso regression coefficients have a closed-form solution that can be computed by solving a linear system of equations.

○ B: Lasso becomes least-squares linear regression when $\lambda = 0$.

○ C: The isocontours of the ℓ_1-regularization term are hypercubes (high-dimensional cubes).

○ D: The main motivation of Lasso is that it tends to select weights of zero for weakly predictive features, which may reduce overfitting and improve interpretability.

(p) [4 pts] In a **convolutional neural network**, which of the following changes to network parameters will **decrease** the size (number of units) of a **hidden layer** of the network?

○ A: Increasing the size of the filters (masks) in a convolutional layer. (Assume we do not use any padding nor "periodic boundaries.")

○ B: Increasing the number of filters (masks) in a convolutional layer.

○ C: Changing a pooling layer to use a 4×4 sliding window with stride 4 instead of a 2×2 sliding window with stride 2.

○ D: Decreasing the size of the mini-batch used for stochastic gradient descent.

(q) [4 pts] Consider a matrix $X \in \mathbb{R}^{n \times d}$ with a singular value decomposition (SVD) $X = UDV^\top$. Which of the following formulae are **equal to the Moore–Penrose pseudoinverse** X^+? (Check a box only if it works for every X.)

○ A: $VD^{-1}U^\top$

○ B: $VD^+U^\top$

○ C: $VD^+DD^+U^\top$

○ D: $VV^\top VD^+U^\top UU^\top$

(r) [4 pts] Below are four examples where we have written a line of pseudocode from a primal, unfeaturized learning algorithm followed by the equivalent line in a dualized, featurized, kernelized version of the same learning algorithm. The primal algorithm computes the weight vector $w \in \mathbb{R}^d$, and the kernelized algorithm computes the dual weight vector $a \in \mathbb{R}^n$. The inputs are the design matrix $X \in \mathbb{R}^{n \times d}$ and a vector of labels $y \in \mathbb{R}^n$. The kernel matrix is $K \in \mathbb{R}^{n \times n}$ and the kernel function is $k(\cdot, \cdot)$. Select the cases where **the kernelized version correctly duplicates the effect of the primal version**.

○ A: primal: $w \leftarrow w + X^\top v$
kernelized: $a \leftarrow a + v$

○ B: primal: $h(z) \leftarrow s(w^\top z)$
kernelized: $h(z) \leftarrow s\left(\sum_{i=1}^{n} a_i \, k(X_i, z)\right)$

○ C: primal: $w \leftarrow w + \epsilon X^\top (y - s(X w))$
kernelized: $a \leftarrow a + \epsilon (y - s(Ka))$

○ D: primal: $\tau \leftarrow \|w\|^2$
kernelized: $\tau \leftarrow a^\top K^2 a$

Extra space: if you need extra space for your answer to a written problem on pages 6–12, you may write here. **Be sure to write "see page 5" under the unfinished answer!**

Q2. [18 pts] Decision Tree Construction

You're investigating the habits of UC Berkeley students. You collect data about whether students climb, drink coffee, and are night owls. You want to use those habits and a decision tree to predict if students are taking CS 189. Each training student is labeled with one of two classes: 189 or MissingOut (not taking 189). Here is the training data you've gathered. (Note that the student numbers are not features and you can't split on them.)

	Climbs	Drinks Coffee	Night Owl	Taking CS 189 (label)
Student 1	Yes	Yes	No	189
Student 2	No	No	Yes	189
Student 3	Yes	No	Yes	MissingOut
Student 4	No	No	No	MissingOut
Student 5	No	No	Yes	189
Student 6	No	Yes	Yes	189
Student 7	Yes	No	Yes	MissingOut
Student 8	Yes	Yes	No	189

(a) [4 pts] **What is the entropy H at the root** of the tree? Your expression can contain logarithms and fractions.

(b) [6 pts] **Which feature should you split on at the root to maximize the information gain? Write an expression for the information gain of the best split.** Your expression can contain logarithms and fractions, but **simplify** the fractions as much as possible. Show your work.

(c) [4 pts] **Draw the complete decision tree** that maximizes the information gain at each split. In each leaf, write

- what class is assigned (189 or MissingOut) and
- which training points reach that leaf (numbered 1–8).

For each internal treenode, write

- the splitting feature and
- which child is "Yes" and which child is "No."

Split until all the leaves are pure. Do not write any entropies or information gains.

(d) [4 pts] Suppose that we are building a random forest with this training set, but we do **not** use bagging and every tree receives the full training set; the only randomization we use is randomized selection of features at each treenode (in the usual way taught in class). After training the forest, you discover that the first tree in your ensemble classifies a test point in which all three features are "Yes" as MissingOut. By contrast, your tree in part (c) (if it's correct) predicts 189.

Explain what must have happened during training that led to the tree in the random forest making a different prediction than your tree from part (c).

Q3. [20 pts] Backpropagation

Consider a fully-connected neural network with d units in the input layer, one hidden layer of k units with ReLU activations, and a single output unit with a sigmoid activation. (Hence there are two layers of weight connections.) We are using the network for two-class classification; we predict in-class for $z > 0.5$ or out-of-class for $z \le 0.5$. Each training point $X_i \in \mathbb{R}^d$ is accompanied by a ground truth label y_i equal to either 1 (in class) or 0 (not in class). We train our model with stochastic gradient descent (one training point at a time; no batches or mini-batches) and the logistic loss $L(z, y)$ (for prediction z and label y).

Let $V \in \mathbb{R}^{k\times d}$ be the weight matrix associated with the first layer of connections. As there is only one output unit, we represent the weights in the second layer of connections as a vector $w \in \mathbb{R}^k$. For simplicity, we do not use bias terms. Here is pseudocode for the forward pass with training point $x \in \mathbb{R}^d$, label $y \in \{0, 1\}$, hidden layer $h \in \mathbb{R}^k$, and output $z \in \mathbb{R}$.

```
$h \leftarrow r(Vx)$                              hidden units; $r$ is the ramp function (a ReLU) applied element-wise to the vector $Vx$
$z \leftarrow s(w^\top h)$                        $z$ is the output unit; $s(\gamma) = 1/(1 + e^{-\gamma})$ is the sigmoid/logistic function
$L \leftarrow -y \ln z - (1 - y)\ln(1 - z)$       compute the logistic loss
```

Work out the equations for backpropagation in this neural network. The derivatives and gradients we ask you to derive may include x, h, z, w, V, y, and/or components of those vectors and matrix. Your final formulae may **not** include s nor s'; express derivatives related to the sigmoid function in terms of x, h, z, w, V, and/or y. All vectors and gradients are column vectors.

(a) [2 pts] **Write the derivative r' of the ramp function** $r(\gamma)$ (where γ is a scalar real value). To make backpropagation work, choose a reasonable value for r' at $\gamma = 0$ (rather than the actual derivative).

(b) [3 pts] Derive $\frac{\partial L}{\partial z}$.

(c) [3 pts] **Derive $\frac{\partial L}{\partial h_i}$ as a function of $\frac{\partial L}{\partial z}$**, as well as x, h, z, w, and/or V. (Do not substitute your expression from part (b) yet; you'll do that in part (d).) **Show your work.** We want to see s' in an intermediate step, but **not** in your final answer.

(d) [3 pts] Substitute your expression for $\frac{\partial L}{\partial z}$ from part (b) into your expression for $\frac{\partial L}{\partial h_i}$ from part (c) and **simplify as much as possible.** Show your work. Hint: there should be no need for a fraction in your final answer.

(e) [2 pts] **Derive $\nabla_w L$ as a function of $\frac{\partial L}{\partial z}$.** Then **substitute in your expression for $\frac{\partial L}{\partial z}$ from part (b) and simplify** as much as possible. Again, do not have s' in your final answer. (There's no need to show your work this time.)

(f) [2 pts] Let $V_i^\top$ denote the ith row of V (so V_i is a column vector). **Derive $\nabla_{V_i} L$ as a function of** $\frac{\partial L}{\partial h_i}$, as well as x, h, z, w, and/or V. Your final answer may have r' in it.

(g) [3 pts] **Write pseudocode to perform a pass of backpropagation** (following the forward pass we wrote above) and **a step of stochastic gradient descent** with learning rate (step size) ϵ, given a single training point x and a label y. Do **not** directly compute $\frac{\partial L}{\partial z}$, as we want to take advantage of the simplifications in parts (d) and (e).

(h) [2 pts] Suppose we run **batch gradient descent** to convergence with this neural network, so V and w are at **a local minimum of the cost function**, which is the mean of the training points' loss functions. After the network is trained, let $H_i \in \mathbb{R}^k$ be the value of the vector of hidden units when the training point X_i is presented at the inputs of the neural network. Suppose we run logistic regression with no bias term (no α) on the points $H_1, H_2, \ldots, H_n$ with the labels $y_1, y_2, \ldots, y_n$, obtaining a weight vector w_{logreg}. **What is the relationship between the final neural network weights V and w and the final logistic regression weights w_{logreg}?**

Q4. [14 pts] The Bayes Classifier

We have a distribution $\mathcal{D}$ of real numbers with class labels. When we draw a random point-and-label $(X, Y) \sim \mathcal{D}$, There is a 40% chance that the label Y is class A, in which case X is a random real number drawn from a uniform distribution on the interval $[-1, 1]$. There is a 60% chance that the label Y is class B, in which case X is a random real number drawn from a distribution on $[-1, 1]$ with the probability density function (PDF) $f_{X|Y=\mathrm{B}}(x) = 1 - |x|$. We will use the 0-1 loss function.

(a) [6 pts] **What is the Bayes optimal decision boundary? Show your work.** (Hint: it might help you to draw a plot.)

(b) [4 pts] **What is the Bayes decision rule** $r^*(x)$**?** (Over the domain $[-1, 1]$; don't worry about points outside that range.)

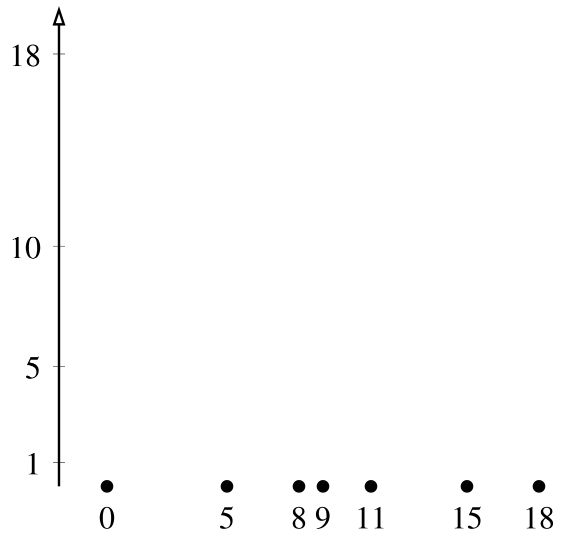
(c) [4 pts] **What is the Bayes risk** $R(r^*)$**? Show your work and express your answer as an exact fraction.** (If you drew a plot, it should help you here too.)

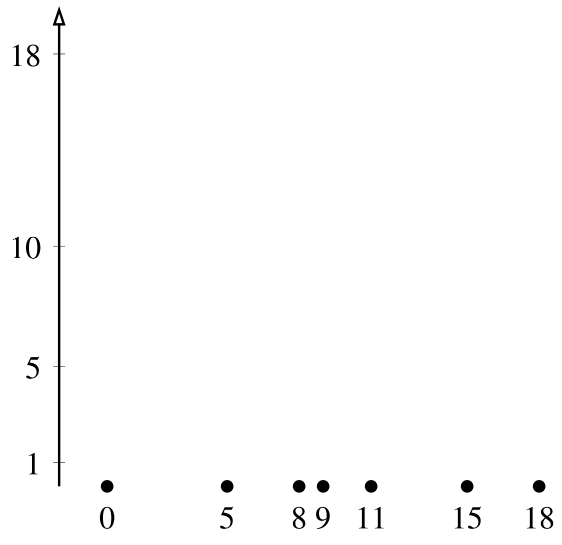
Q5. [12 pts] CS 189/289A Art School

(a) [6 pts] Consider a range space $(\mathbb{R}^2, H)$ where H is the set of all lines in the plane. (Not halfplanes; lines! Not line segments; infinite lines! A point $x \in \mathbb{R}^2$ is in class C if x lies precisely on the line; otherwise it's not in class C. Yes, it's not a very useful classifier.)

Consider a set X of **three collinear** (but distinct) points in the plane. (Collinear means they all lie on a common line.) How many dichotomies of X can you obtain with the hypothesis class H? **Draw all of the dichotomies:** for each dichotomy, draw the three points and a hypothesis that generates the dichotomy.

(b) [6 pts] Let's do hierarchical clustering in one dimension. Specifically, we will do agglomerative clustering with **complete linkage** (which you might think of as the "max" linkage) on the numbers listed below. **Draw the dendrogram** for the points $\{0, 5, 8, 9, 11, 15, 18\}$. The distance function for two individual points p and q is $d(p, q) = |p - q|$. Make sure you draw the horizontal bars at the correct heights! We have included two copies of the figure in case your first drawing is wrong and you need to start over, but you only need to draw one.





Q6. [14 pts] Random Projection

Recall the method of random projection, where we project every sample point $x \in \mathbb{R}^d$ onto a randomly chosen k-dimensional subspace $S \subset \mathbb{R}^d$, where $k < d$. Let's consider a closely related method for dimensionality reduction that also approximately preserves distances between points (with high probability).

(a) [4 pts] Recall from our discussion of principal components analysis (PCA) that, if two conditions hold, we can orthogonally project a point $x \in \mathbb{R}^d$ onto the subspace spanned by the vectors $v_1, \ldots, v_k$ with the formula $\tilde{x} = \sum_{i=1}^{k} (x \cdot v_i)\, v_i$. **What two conditions** must $v_1, \ldots, v_k$ satisfy for this formula to be correct?

(b) [5 pts] Consider a random matrix $G \in \mathbb{R}^{k \times d}$, with $k < d$. Every component of G is a random real number $G_{ij} \sim \mathcal{N}(0, 1/d)$; that is, every component is drawn from a univariate normal distribution with mean zero and variance $1/d$. These components are all independent of each other. Consider a (**not** random) vector $x \in \mathbb{R}^d$. **Show that** the expected squared ℓ_2-norm of Gx is

$$E\left[\|Gx\|^2\right] = \frac{k}{d}\|x\|^2.$$

(c) [5 pts] The result in part (b) suggests that if we choose a random G as described in part (b) and replace the sample points $X_1, X_2, \ldots, X_n \in \mathbb{R}^d$ with the k-dimensional points

$$\tilde{X}_1 = \sqrt{\frac{d}{k}} G X_1, \quad \tilde{X}_2 = \sqrt{\frac{d}{k}} G X_2, \quad \ldots, \quad \tilde{X}_n = \sqrt{\frac{d}{k}} G X_n,$$

the distance between two points might not "change much"; that is, $\|\tilde{X}_i - \tilde{X}_j\|$ is an approximation of $\|X_i - X_j\|$. This is similar to computing an orthogonal projection onto a randomly chosen k-dimensional subspace, but not quite the same.

In the limit as $d \to \infty$ (with k fixed), **why does this method become functionally the same as the method of random projection?** Hint: what two counterintuitive properties of vectors in high-dimensional spaces are relevant, and which vectors should we apply them to? (A written answer suffices; no math is necessary.)