1. This question will make sure that you understand complex numbers and complex arithmetic, and that they are very similar to real numbers and real arithmetic in the properties they satisfy.

   In particular, it is a fact that the set of *real numbers* satisfies the following familiar set of rules, also called the axioms defining a *field*:

   (a) Commutativity of addition: $a + b = b + a$ for all a and b.
   (b) Commutativity of multiplication: $a * b = b * a$ for all a and b.
   (c) Associativity of addition: $a + (b + c) = (a + b) + c$ for all a, b, c.
   (d) Associativity of multiplication: $a * (b * c) = (a * b) * c$ for all a, b, c.
   (e) Distributivity: $a * (b + c) = (a * b) + (a * c)$ for all a, b, c.
   (f) Zero exists: there is a special number called 0 satisfying $a + 0 = a$ for all a.
   (g) Negative numbers exist: for each a there is a number called $-a$ satisfying $a+(-a) = 0$.
   (h) One exists: there is a special number called 1 satisfying $a * 1 = a$ for all a.
   (i) Reciprocals exist: for each $a \neq 0$ there is a number called $\frac{1}{a}$ satisfying $a * (1/a) = 1$.

   It turns out that the complex numbers, written $a + i * b$ with a and b real, and $i = \sqrt{-1}$, with addition and multiplication defined by

   - $(a + i * b) + (c + i * d) = (a + c) + i * (b + d)$.
   - $(a + i * b) * (c + i * d) = (a * c - b * d) + i * (a * d + b * c)$.

   also form a field, i.e. they satisfy the above axioms.

   Your assignment is to prove that one of the above properties holds for complex numbers, assuming they hold for real numbers. In particular, you should prove property (i). Make sure that you explicitly show what $1/(x + i * y)$ is, where $a = x + i \cdot y$.

2. This question explores another way to write complex numbers that you will need to know besides the usual one ($z = a + i \cdot b$, $i = \sqrt{-1}$, a and b real). This way is called *polar form*. One can write the nonzero complex number $z = a + i * b$ as

$$a + i * b = \rho * (\frac{a}{\rho} + i * \frac{b}{\rho}) \quad \text{where} \quad \rho = \sqrt{a^2 + b^2} \ .$$

   ρ is also called the *magnitude* or *absolute value* of the complex number, and also written $\rho = |z|$. Let $c = \frac{a}{\rho}$ and $s = \frac{b}{\rho}$. Since $c^2 + s^2 = 1$, we can think of c and s as the cosine and sine of some angle $\theta = \arccos(c) = \arcsin(s)$, i.e. $c = \cos\theta$ and $s = \sin\theta$. This lets us write $z = \rho(\cos\theta + i * \sin\theta)$. θ is called the *argument* of z.

   Show that $\cos\theta + i * \sin\theta = e^{i\theta}$, by considering the Taylor series expansions of both sides. You may use the known forms of the Taylor expansions of e^x, $\sin x$ and $\cos x$. $z = \rho e^{i\theta}$ is called the *polar form* of z.

3. The *complex conjugate* of $z = a + i \cdot b$ is $\bar{z} = a - i \cdot b$. Let $z = \rho e^{i\theta}$ be the polar form of z. Then the following are facts about the complex conjugate:

   (a) $\bar{z} = \rho e^{-i\theta}$

(b) $z\bar{z} = \rho^2$

(c) $1/z = \bar{z}/|z|^2$.

(d) $z = z_1 \pm z_2$ implies $\bar{z} = \bar{z_1} \pm \bar{z_2}$

(e) $z = z_1 \cdot z_2$ implies $\bar{z} = \bar{z_1} \cdot \bar{z_2}$

(f) $z = z_1/z_2$ implies $\bar{z} = \bar{z_1}/\bar{z_2}$

Prove only parts a), b), and c).

4. *The Complex Plane.*

   (a) Draw a picture with the vectors for the complex numbers $z_1 = i$, $z_2 = 1 + i$, $\bar{z_2}$, $z_3 = (1 + i)/\sqrt{2}$, $z_1 + z_2$, $z_2 * z_3$.

   (b) If z_1 and z_2 are any complex numbers, show that the complex number $z = z_1 + z_2$ is represented by the vector gotten by *adding the vectors* for z_1 and z_2, i.e. putting the tail of the vector for z_2 at the head of the vector for z_1, and taking its head.

   (c) If z_1 and z_2 are any complex numbers, show that the complex number $z = z_1 * z_2$ is represented by the following vector: its length is the product of the lengths of the vectors for z_1 and z_2, and its argument is the sum of the arguments of z_1 and z_2.

5. (From the Spring 98 midterm.) In class we derived the FFT for vectors of length n a power of two. In this question we will derive the FFT for $n = 3^s$, a power of three. (More generally, it is possible to compute the FFT efficiently whenever n is a product of many small primes, $n = 2^j 3^k 5^l \ldots$, but we will not pursue this generality here.)

   (a) Let $p(z) = \sum_{j=0}^{n-1} p_j \cdot z^j$ be a polynomial of degree at most $n - 1$, where $n = 3^s$. Show that $p(z)$ can be written as the sum

   $$p(z) = p0(z^3) + z \cdot p1(z^3) + z^2 \cdot p2(z^3) \qquad (1)$$

   where $p0(z')$, $p1(z')$ and $p2(z')$ are each polynomials of degree at most $(n/3) - 1$. Be sure to explicitly exhibit the coefficients of each polynomial.

   (b) Let $\omega = e^{2\pi i/n}$, $i = \sqrt{-1}$, be a primitive n-th root of unity. Using equation (1), show that you can evaluate $p(z)$ at the n points ω^0, ω^1, ω^2, ... ,ω^{n-1}, *given* the values of the 3 polynomials $p0(z')$, $p1(z')$ and $p2(z')$ at the $n/3$ points ω^0, ω^3, ω^6, ω^9, ... , ω^{n-3}. You should write down a loop that evaluates $p'_j = p(\omega^j)$, for $j = 0$ to $n - 1$, in terms of the values of $p0(z')$, $p1(z')$ and $p2(z')$.

   (c) Write a recursive subroutine for evaluating $p(z)$ at ω^j, $j = 0, ..., n - 1$. Use your answer from the previous part in your answer.

   (d) What is the complexity of your recursive subroutine? You should write down a recurrence for the complexity $T(n)$, justify it, and quote a theorem from class to solve it.

6. CLR 32-4. For part d you may assume that n is a power of 2. (There is a typo in the book: $Q_{ij}(x)$ should be defined as $Q_{ij}(x) = A(x) \bmod P_{ij}(x)$.)