Non-Cooperative Games

in an uncertain environment

Roger J-B Wets

rjwets@ucdavis.edu

University of California, Davis
I. Deterministic Version
Finding a Nash-equilibrium

- problem formulation
- the Nash-function associated with a game
- max-inf points and Nash equilibrium points
- remarks about computational schemes
Agent’s Problem

- agents: \( a \in \mathcal{A}, \ |\mathcal{A}| \text{ finite} \ (\text{two?}) \)
- \( x_a \in C_a \subset \mathbb{R}^{n_a} \), decision of agent ‘\( a \)’
- \( x_{-a} \in \mathbb{R}^{N-n_a} \), decisions of all other agents
- \( u_a(x_a, x_{-a}) : \mathbb{R}^N \rightarrow \mathbb{R}, \ a\text{-performance fcn} \)

Nash Equilibrium: \( \bar{x} = (\bar{x}_a \ a \in \mathcal{A}) \in \mathbb{R}^N \)
such that for all \( a \in \mathcal{A} \):

\[
\bar{x}_a \in \text{argmin} \left\{ u_a(x_a, \bar{x}_{-a}) \mid x_a \in C_a \right\}
\]
Mathematical Model

\[ \begin{align*} 
\max \sum_{j \in J} u_{a,j} (x_{a,j} - z_{-a,j}) \\
\text{so that} \sum_{j \in J} x_{a,j} & \leq 1, \quad 0 \leq x_{a,j} \leq u b_{a,j} \\
\sum_{j \in J} t_{a,j} x_{a,j} & \leq T_{\text{capcy}} 
\end{align*} \]
$\alpha$-performance function

\[ u_{a,j} \]

\[ X_{a-Z-a} \]
The Nash-function

\[ N(x, y) = \sum_{a \in A} u_a(x_a, x_{-a}) - u_a(y_a, x_{-a}) \]

if \( \forall a \in A : x_a \in C_a, y_a \in C_a \)

= \(-\infty\) if \( \forall a \in A, y_a \in C_a \)

and \( x_a \notin C_a \) for some \( a \in A \)

= \( \infty \) otherwise
Equilibrium and max-inf points

\( \bar{x} = (\bar{x}_a, a \in A) \) with \( \bar{x}_a \in C_a \) is a Nash equilibrium

if and only if

\[
\max_{x:(x_a, a \in A)} \inf_{y:(y_a, a \in A)} N(x, y) = \inf_{y:(y_a, a \in A)} N(\bar{x}, y) \geq 0,
\]

i.e., \( \bar{x} \) is an argmax-inf point of the Nash-function.

\[
N(x, y) \approx \sum_{a \in A} u_a(x_a, x_{-a}) - u_a(y_a, x_{-a})
\]
Existence of Nash Equilibrium

\[ N(x, y) \approx \sum_{a \in A} u_a(x_a, x_{-a}) - u_a(y_a, x_{-a}) \]

- existence \( \iff \exists \bar{x} \in \text{argmax-inf } N \).
- if \( N \) is usc in \( x \), convex in \( y \) \( \Rightarrow \) existence
- \( N \) usc in \( x \) if \( \forall a, u_a \) is usc \& \( u_a(x_a, \cdot) \) is lsc
- \( N \) convex in \( y \) if \( C_a \) convex \& \( u_a(\cdot, x_{-a}) \) concave.
Nash equilibrium
Stability of Nash Equilibrium

\[ N(x, y) \approx \sum_{a \in A} u_a(x_a, x_{-a}) - u_a(y_a, x_{-a}) \]

- stability of Nash Equilibrium = stability of argmax-inf of Nash-function

- \( C^{\nu}_a \rightarrow C_a \) & \( u^{\nu}_a \Rightarrow u_a \Rightarrow \text{lopsided convergence of Nash-functions } N^{\nu}. \)

- lopsided convergence of Nash-fcns \( N^{\nu} \Rightarrow \) convergence of argmax-inf points!

lopsided convergence of bivariante functions

is related to epi-convergence of (univariate) functions
Augmented Nash-function

- PL-homotopy methods, low dimensional
- optimization-based method via augmentation

\[ \mathcal{N}_r(x, y) = \left\{ \begin{array}{l}
\inf_u \left\{ N(x, u) + r\|u\| - \langle y, u \rangle \right\} \\
\sup_z \left\{ N(x, z) \right\} \quad \|z - y\|_o \leq r \end{array} \right. \]

where \( \| \cdot \| \) and \( \| \cdot \|^o \) are dual norms.


**Iterations**

Set $C = \prod_{a \in A} C_a$.

$$N(x, y) = \sum_{a \in A} u_a(x_a, x_{-a}) - u_a(y_a, x_{-a}) \text{on } C \times C$$

$$\tilde{N}_r(x, y) = \sup_{z} \{ N(x, z) \mid \|z - y\|^o \leq r \}$$

\[
y^{k+1} = \arg\max_{y \in C} \left\{ \sup_{z} N(x^k, z) \mid \|z - y\|^o \leq r_k \right\}
\]

\[
x^{k+1} = \arg\min_{x \in C} \left\{ \sup_{z} N(x, z) \mid \|z - y^{k+1}\|^o \leq r_{k+1} \right\}
\]

as $r_k \uparrow \infty$, $x^k \to \bar{x}$ max-inf point of $N$. 

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II. Stochastic Environment
Existence, Algorithms

- non-cooperative and uncertain environment
- the agents’ optimization problems
- information flow & non-anticipativity
- disintegration of the ‘stochastic’ problem
- Nash-fcns associated with a stochastic game
- remarks about existence, computational procedures
Formulation

- $\xi \subseteq \Xi$ uncertain (stochastic) environment
- $x_a^1 \in C_a^1$ decision of agent ‘$a$’ @ time 1 (now)
- $x_a^2 \in C_a^2(\xi; x^1)$ decision of agent ‘$a$’ @ time 2
- $u_a^1(x_a^1, x_{-a}^1) + E^a \{u_a^2(\xi; x_a^2(\xi), x_{-a}^2(\xi)) \}$ performance estimate of agent ‘$a$’

Nash Equilibrium: $(\bar{x}_a^1, \bar{x}_a^2(\cdot))$ such that for all $a \in \mathcal{A}$:

$$((\bar{x}_a^1, \bar{x}_a^2(\cdot))) \in \text{argmax } a\text{-performance estimate}$$
Agent’s Problem

\[
\max_{x^1_a, x^2_a(.)} u^1_a(x^1_a, \hat{x}^1_{-a}) + E^a \{ u^2_a(\xi; x^2_a(\xi), \hat{x}^2_{-a}(\xi)) \}
\]

so that

\[
x^1_a \in C^1_a
\]

\[
x^2_a(\xi) \in C^2_a(\xi; x^1_a, \hat{x}^1_{-a}), \ \forall \xi \in \Xi
\]

Two-stage stochastic optimization problem (generalizes to \(n\)-stage, dynamically)

Note: distribution of \(u\) doesn’t depend on but the distribution of the state of the system does
Agent’s Problem

\[
\max_{x_a^1, x_{a^2}(\cdot)} u_a^1(x_a^1, \hat{x}_{-a}^1) + E^a \{ u_a^2(\xi; x_a^2(\xi), \hat{x}_{-a}^2(\xi)) \}
\]

so that

\[
x_a^1 \in C_a^1
\]

\[
x_a^2(\xi) \in C_a^2(\xi; x_a^1, \hat{x}_{-a}^1), \quad \forall \xi \in \Xi
\]

a two-stage stochastic optimization problem
(generalizes to \(N\)-stage, dynamically)
Agent’s Problem

\[
\max_{x^1_a, x^2_a(\cdot)} u^1_a(x^1_a, \hat{x}^1_{-a}) + E^a \{ u^2_a(\xi; x^2_a(\xi), \hat{x}^2_{-a}(\xi)) \}
\]

so that \( x^1_a \in C^1_a \)

\( x^2_a(\xi) \in C^2_a(\xi; x^1_a, \hat{x}^1_{-a}), \ \forall \xi \in \Xi \)

a two-stage stochastic optimization problem
(generalizes to \( N \)-stage, dynamically)

Note: distribution of \( \xi \) doesn’t depend on \( x^1 \)
but the distribution of the state of the system does
Stochastic Optimization

$$\max_{x^1, x^2(\cdot)} u^1(x^1) + E\{u^2(\xi; x^2(\xi))\}$$

so that

$$x^1 \in C^1$$

$$x^2(\xi) \in C^2(\xi; x^1), \ \forall \xi \in \Xi$$

Decision process:

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decision \ x^1 \rightarrow \ observation \ \xi \rightarrow \ recourse \ x^2
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Information process:

- no information about the future is available to $x^1$
- $x^2$ can totally depend on realization $\xi$

i.e., there is a **non-anticipativity restriction** on $x^1$. 

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‘Removing’ non-anticipativity:

$$\max_{x^1(\cdot), x^2(\cdot)} E\{u^1(x^1(\xi)) + u^2(\xi; x^2(\xi))\}$$

so that $$x^1(\xi) \in C^1, \ x^2(\xi) \in C^2(\xi; x^1), \ \forall \xi \in \Xi$$

$$x^1(\xi) = x^1(\xi'), \ \forall \xi, \xi' \in \Xi.$$ 

With a constraint qualification, $$\exists$$ multipliers $$w(\cdot) : \Xi \to \mathbb{R}^{n_1}$$ such that $$E\{w(\xi)\} = 0$$ and

$$\max_{x^1, x^2} E\{u^1(x^1(\xi)) - \langle w(\xi), x^1(\xi) \rangle + u^2(\xi; x^2(\xi))\}$$

so that $$x^1(\xi) \in C^1, \ x^2(\xi) \in C^2(\xi; x^1), \ \forall \xi \in \Xi$$

has the same solution with $$\bar{x}^1(\cdot) = \text{constant.}$$
DISINTEGRATION

One can solve:

$$\max_{x^1(\cdot), x^2(\cdot)} E\{u^1(x^1(\xi)) - \langle w(\xi), x^1(\xi) \rangle + u^2(\xi; x^2(\xi))\}$$

so that $x^1(\xi) \in C^1, x^2(\xi) \in C^2(\xi; x^1), \forall \xi \in \Xi$

by solving for each $\xi \in \Xi$:

$$\max_{x^1, x^2} u^1(x^1) - \langle w(\xi), x^1 \rangle + u^2(\xi; x^2)$$

so that $x^1 \in C^1, x^2 \in C^2(\xi; x^1)$

with $x^1 \in \mathbb{R}^{n_1}, x^2 \in \mathbb{R}^{n_2}$,
Progressive hedging algorithm

Step 0. pick $w^0(\cdot)$ with $E\{w^0(\xi)\} = 0$, $\rho > 0$. 
Progressive hedging algorithm

**Step 0.** pick $w^0(\cdot)$ with $E\{w^0(\xi)\} = 0$, $\rho > 0$.

**Step 1.** for each $\xi \in \Xi$, find $(x^{1,k}(\xi), x^{2,k}(\xi))$ in

$$\max_{x^1 \in C^1, x^2 \in C^2(\xi; x^1)} u^1(x^1) - \langle w^k(\xi), x^1 \rangle + u^2(\xi; x^2)$$
Progressive hedging algorithm

**Step 0.** pick $w^0(\cdot)$ with $E\{w^0(\xi)\} = 0$, $\rho > 0$.

**Step 1.** for each $\xi \in \Xi$, find $(x^{1,k}(\xi), x^{2,k}(\xi))$ in

$$
\max_{x^1 \in C^1, x^2 \in C^2(\xi;x^1)} u^1(x^1) - \langle w^k(\xi), x^1 \rangle + u^2(\xi; x^2)
$$

**Step 2.** set $\bar{x}^{1,k} = E\{x^{1,k}(\xi)\}$

- Stop if $\max_{\xi \in \Xi} \| x^{1,k}(\xi) - \bar{x}^{1,k} \| < \varepsilon$
- otherwise, $w^{k+1}(\xi) = w^k(\xi) + \rho(x^{1,k}(\xi) - \bar{x}^{1,k})$
  and return to **Step 1.** with $k = k + 1$
Progressive hedging algorithm

**Step 0.** pick \( w^0(\cdot) \) with \( E\{w^0(\xi)\} = 0, \rho > 0 \).

**Step 1.** for each \( \xi \in \Xi \), find \((x^{1,k}(\xi), x^{2,k}(\xi))\) in

\[
\max_{x^1 \in C^1, x^2 \in C^2(\xi; x^1)} u^1(x^1) - \langle w^k(\xi), x^1 \rangle + u^2(\xi; x^2)
\]

**Step 2.** set \( \bar{x}^{1,k} = E\{x^{1,k}(\xi)\} \)

- Stop if \( \max_{\xi \in \Xi} \|x^{1,k}(\xi) - \bar{x}^{1,k}\| < \varepsilon \)
- otherwise, \( w^{k+1}(\xi) = w^k(\xi) + \rho(x^{1,k}(\xi) - \bar{x}^{1,k}) \)
and return to **Step 1.** with \( k = k + 1 \)

**Convergence**

- add a proximal term \(- (\rho/2) \| x^1 - \bar{x}^{1,k-1} \|^2 \)
- linear convergence in \((\bar{x}^{1,k}, w^k)\)
Disintegrated equilibrium: $\forall \xi$

for $a \in \mathcal{A}$, let $w_a : \Xi \rightarrow \mathbb{R}^{n_a}$; $w = (w_a, a \in \mathcal{A})$

$$E^a \{w_a(\xi)\} = 0$$

for each $\xi \in \Xi$, find $(\bar{x}^1, \bar{x}^2(\xi))$ the deterministic Nash equilibrium when the agents’ problems are:

$$\max_{x_a^1, x_a^2} u_a^1(x_a^1, \hat{x}_{-a}^1) - \langle w_a(\xi), x_a^1 \rangle + u^2(\xi; x_a^2, \hat{x}_{-a}^2(\xi))$$

so that $x_a^1 \in C_a^1$, $x_a^2 \in C_a^2(\xi; x_a^1, \hat{x}_{-a}^1)$
Existence and Algorithm(s)

Existence:

- define Nash-functions for disintegrated problems (=> existence)
- and use stability of Nash equilibrium w.r.t. perturbations ($u^1 - \langle w, \cdot \rangle$)

Solution Procedure:

- overall strategy of the Progressive Hedging algorithm to obtain convergence of the (non-anticipativity) multipliers $w_a$.
- in Step 1 of PHa, iterate on Augmented Nash-function to obtain argmax-inf point.